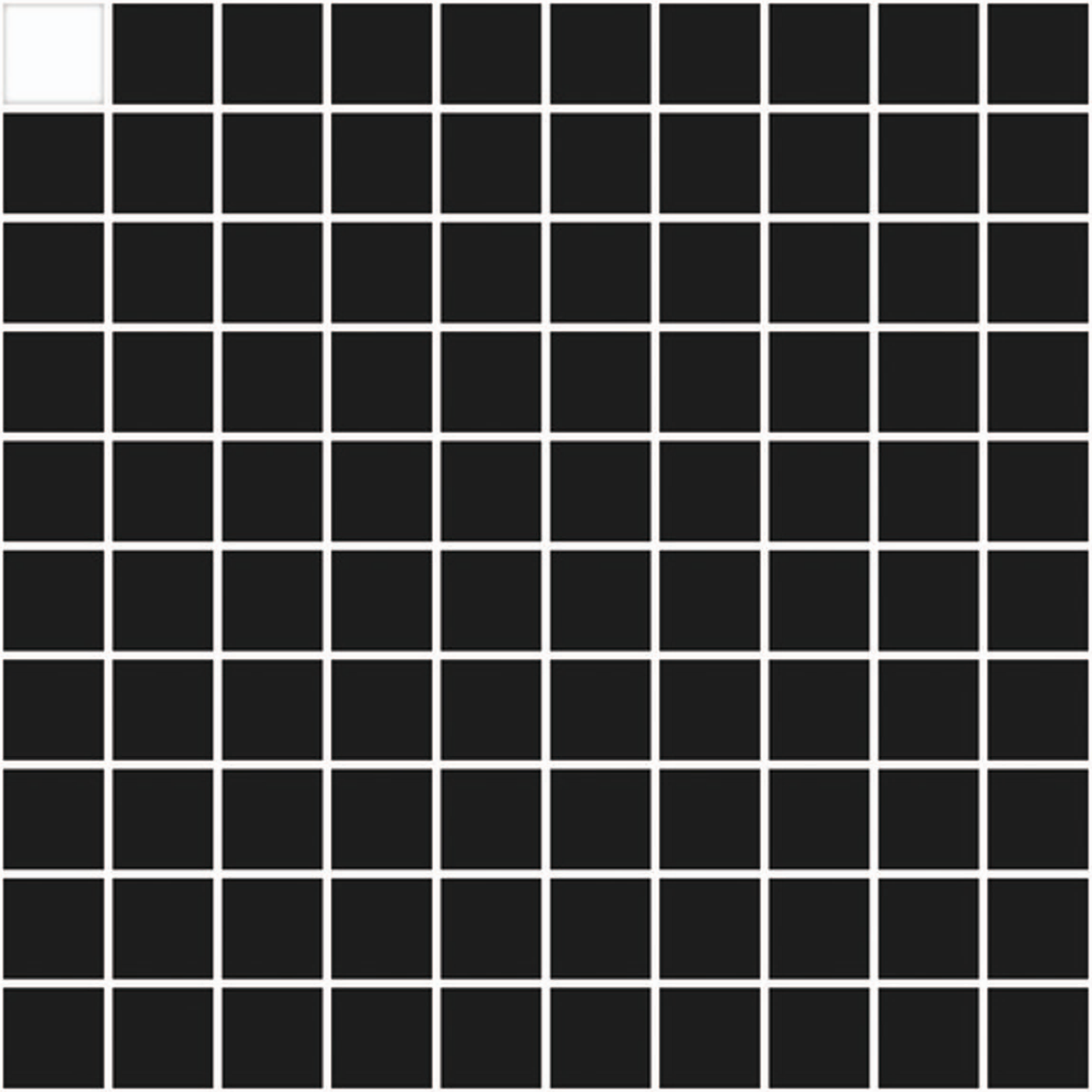


99 Variations on a Proof

Philip Ording

99 Variations on a Proof

99 Variations on a Proof

Philip Ording

Princeton University Press
Princeton and Oxford

Published by Princeton University Press
41 William Street, Princeton, New Jersey 08540
6 Oxford Street, Woodstock, Oxfordshire OX20 1TR

press.princeton.edu

LCCN 2018942718
ISBN 978-0-691-15883-9

British Library Cataloging-in-Publication Data is available

Editorial: Vickie Kearn, Susannah Shoemaker, and Lauren Bucca
Production Editorial: Nathan Carr
Cover and Text Design: Chris Ferrante
Production: Jacquie Poirier
Publicity: Sara Henning-Stout and Kathryn Stevens
Copyeditor: Lor Campbell Gehret

This book has been composed in STIX Fonts and Replica

Printed on acid-free paper ∞

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

For Alexandra, who never
said it couldn’t be done.

Contents

Preface ix

0 Omitted 1

1 One-Line 3

2 Two-Column 5

3 Illustrated 7

4 Elementary 9

10 Wordless 23

11 Exam 25

12 Ruler and Compass 27

13 Reductio ad Absurdum 29

14 Contrapositive 31

20 Definitional 43

21 Blackboard 47

22 Substitution 49

23 Symmetry 51

24 Another Symmetry 53

30 Formulaic 69

31 Counterexample 71

32 Another Counterexample 73

33 Calculus 75

34 Medieval 77

40 Induction 91

41 Newsprint 93

42 Analytic 95

43 Screenplay 97

44 Omitted with Condescension 105

50 Chromatic 117

51 Topological 119

52 Antiquity 121

53 Marginalia 125

54 Arborescent 129

60 Geometric 141

61 Modern 143

62 Axonometric 145

63 Back of the Envelope 149

64 Research Seminar 151

70 Another Medieval 163

71 Blog 167

72 Translated 171

73 Another Translated 173

74 Another Interpreted 175

80 Paranoid 191

81 Doggerel 193

82 Inconsistency 195

83 Correspondence 197

84 Tabular 199

90 Retrograde 215

91 Mystical 217

92 Refereed 219

93 Neologism 221

94 Authority 223

5 **Puzzle** 11
6 **Axiomatic** 13
7 **Found** 17
8 **Prerequisite** 19
9 **Monosyllabic** 21

15 **Matrices** 33
16 **Ancient** 35
17 **Interpreted** 37
18 **Indented** 39
19 **Jargon** 41

25 **Open Collaborative** 57
26 **Auditory** 61
27 **Algorithmic** 63
28 **Flow Chart** 65
29 **Model** 67

35 **Typeset** 79
36 **Social Media** 83
37 **Preprint** 85
38 **Parataxis** 87
39 **Origami** 89

45 **Verbal** 107
46 **Cute** 109
47 **Clever** 111
48 **Computer Assisted** 113
49 **Outsider** 115

55 **Prefix** 131
56 **Postfix** 133
57 **Calculator** 135
58 **Inventor's Paradox** 137
59 **Patented** 139

65 **Tea** 153
66 **Hand Waving** 155
67 **Approximate** 157
68 **Word Problem** 159
69 **Statistical** 161

75 **Slide Rule** 181
76 **Experimental** 183
77 **Monte Carlo** 185
78 **Probabilistic** 187
79 **Intuitionist** 189

85 **Exhaustion** 201
86 **Another Substitution** 203
87 **Mechanical** 207
88 **Dialogue** 209
89 **Interior Monologue** 213

95 **First Person** 225
96 **Electrostatic** 227
97 **Psychedelic** 229
98 **Mondegreen** 231
99 **Prescribed** 233

Postscript 235
Acknowledgments 237
Notes 239
Sources 247
Index 257

Preface

On April 19, 1610, upon receiving an advance copy of Galileo's *Starry Messenger*, Johannes Kepler composed a fan letter. "I may perhaps seem rash in accepting your claims so readily with no support of my own experience," Kepler wrote to Galileo. "But why should I not believe a most learned mathematician, whose very style attests the soundness of his judgement?"[1] We are not accustomed today to thinking of a mathematician's work in terms of style. A proof is a form of argument, but the truth of the theorem it proves would hardly seem to depend on any rhetorical, let alone stylistic, features of its proof. The received wisdom is that mathematics, the universal language of science, has one style—*the mathematical style*—characterized by symbolic notation, abstraction, and logical rigor.[2]

This book aims to challenge that conception of mathematics. While a belief in the universality and unity of *ars mathematica* is not without reason, a moment's reflection gives rise to some basic questions. Where did "the" mathematical style come from? How has it developed with the growth of mathematical knowledge? What opportunities does it open or foreclose? How has its potential evolved with changes in the forms of writing and, therefore, ways of reading, mathematics? What are its expressive, cognitive, and imaginative possibilities?

These questions, at heart, concern the *literature* of mathematics. To survey this literature—a vast body of material ranging in subject from algebra to geometry, number theory to physics, logic to statistics, and dating from Babylonian tablets of the Bronze Age to the peer-reviewed journals and electronic preprints of today—is clearly beyond the scope of a book this size. Instead, I will describe a cross section of mathematics using a method inspired by Raymond Queneau's *Exercises in Style*. This literary work from 1947 takes the same simple story—that of a peculiar individual who is first seen in a dispute on a bus, and then later in conversation with a friend about the position of a coat button—and transforms it in ninety-nine different ways. Queneau's stylistic exercises exemplify various forms of prose, poetry, and speech, as well as more striking contortions, such as "Onomatopoeia," "Dog Latin," and "Permutations by Groups of 2, 3, 4, and 5 Letters." Queneau, in addition to being an author and poet, was also an amateur mathematician, and together with the mathematical historian François Le Lionnais he cofounded the experimental writing group known as the Oulipo. The name of the mostly French writing group is an acronym for *Ouvroir de Litérature Potentielle* (Workshop for Potential Literature), and its membership includes writers, artists, and mathematicians such as Georges Perec, Italo Calvino, Marcel Duchamp, Jacques Roubaud, Claude Berge, and Michèle Audin. The stated purpose of the group is to explore the possibilities for literature derived from mathematically inspired rules or constraints.[3] As soon as I learned about the Oulipo and Queneau's book, I wanted to see what effect constrained writing strategies would have on a mathematical narrative—a proof.

The theme I chose for *99 Variations on a Proof* is an algebraic equation known as a cubic equation, and every chapter proves the same minor—some might say trivial—theorem about its solutions. Many proofs, from 16 Ancient to 61 Modern, emerge from the mathematical literature on cubics. In some cases this happened quite directly, the extreme example being 7 Found, which I discovered, ready-made, on a page of the most famous Renaissance treatise on algebra. More often than not, however, variations required considerable interpretation and invention. Sometimes this was because the style originated in a subject area peripheral to cubics, as in 6 Axiomatic or the physics-based proof 96 Electrostatic. Still more distant translations were needed to convey styles outside of mathematics entirely, such as the musical score 26 Auditory and the architectural 62 Axonometric.

Some proofs aim to satisfy a particular standard of rigor, some fall short of today's standard, and some have other aims entirely.

Each variation, with relatively few exceptions, appears on a single page with a brief discussion on the reverse side of the page. The secondary text includes explanatory details, source information, and my comments on the nature and significance of each style. Cross references to related variations invite readers to deviate from the idiosyncratic order that I've given to the chapters and find their own paths through the book.

This is not a mathematical treatise on cubic equations, and my choice of the particular cubic here was made almost arbitrarily. Despite the historical threads implied by the chapter titles, this is not a book of mathematical history; while the ontological status of content and style is a matter of some debate, this is also not a work of philosophy. It is a book *about* mathematics, its attitudes, norms, perspectives, and practices—in short, its culture.[4]

Other comparative studies of mathematical proof have addressed the relationship between content and form in different ways. In 1938, one H. Pétard published "A Contribution to the Mathematical Theory of Big Game Hunting," which offers thirty-eight applications of modern mathematics and physics to the problem of catching a lion.[5] During the writing of *99 Variations on a Proof*, two other mathematical renderings of Queneau's *Exercises* appeared: *Rationnel mon Q* by Ludmila Duchêne and Agnès Leblanc and *Exercises in (Mathematical) Style* by John McCleary. While there is necessarily some overlap between these books, it is surprising that studies of style could themselves vary in style so much. This in and of itself further confirms the potential of the basic premise of Queneau's original.

What distinguished the style of that most learned mathematician, Galileo? "For him, good thinking means quickness, agility in reasoning, economy in argument, but also the use of imaginative examples,"[6] according to Italo Calvino. This Oulipian finds the clearest statement of Galilean style in the following passage of *The Assayer*

from 1623: while criticizing an adversary's reliance on authority to carry an argument, Galileo asserts, "but discoursing is like coursing, not like carrying, and one Barbary courser can go faster than a hundred Frieslands."[7] Calvino calls this Galileo's "declaration of faith—style as a method of thought and as literary taste."[8] This is a faith that I have tried to keep.

My motivation for this project, from beginning to end, has been to try to conceptualize mathematics as a literary or aesthetic medium. There is no shortage of evidence that professional mathematicians describe their work in aesthetic terms, but the terms they use, at least publicly, are very limited. The oft-repeated "beauty" and "elegance" may be important components of mathematical taste, but they fail to convey its range or subtlety or how it relates to literary and aesthetic experiences beyond mathematics.[9] The ninety-nine (or, if you admit an omitted proof the same status as the others, one hundred) proofs serve to highlight the material differences in logic, diction, imagery, and even typesetting that give tone and flavor to mathematics.[10] I hope that readers with little or no predisposition to the subject matter will begin to perceive these stylistic differences by merely paging through examples, stopping to look more closely at proofs that reflect—or offend—their sensibility and moving lightheartedly forward from any that do not. The reader inclined to delve deeper might recognize that the book itself is a mathematical game. In any case, if mathematics is made more vivid as a result of it passing through the reader's hands, the book will have served its intended purpose.

0

Omitted

Theorem. *If* $x^3 - 6x^2 + 11x - 6 = 2x - 2$, *then* $x = 1$ *or* $x = 4$.

Proof. Omitted. $\square$

Why bother with proof at all? Proofs are explicitly omitted by authors for a variety of reasons, one being aesthetic. In a standard textbook of undergraduate level abstract algebra, one finds: “The proof of this proposition is notationally unpleasant without having any interesting features, so we omit it.”[11] Readers often forgive such omissions in expository work; still, one has to be careful taking mathematics on faith.

It is worth noting a few peculiarities of the proposition before embarking on its proof, notationally pleasant or not.

We are given an algebraic equation in terms of some numbers, an unknown variable x, its square x^2, and its cube x^3. This makes it a *degree-three polynomial* equation or, simply, a *cubic* equation. A more standard form for the equation would organize all the terms to one side of the equals sign as $x^3-6x^2+9x-4=0$. That’s the first peculiarity, and several proofs will take this normalizing step as a point of departure.

Actually, that’s the second peculiarity—the *first* peculiarity of this theorem is that it doesn’t state what x *is*. Some mathematical readers will tolerate a skipped proof, but the introduction of a variable without specifying its domain is a universally condemned sin of omission. Why? Because it invites ambiguity. For the purposes of our exploration of style, however, it will function as a highly productive ambiguity, to borrow a term from the American philosopher and poet Emily Grosholz.[12]

A final, more mathematically interesting, feature is that the cubic equation has just two solutions. If you haven’t forgotten the quadratic formula and its $\pm$ sign, you may recall that degree-two equations have two roots. Though I have yet to meet anyone who has committed it to memory, there is a *cubic formula* (see 30 Formulaic), and it extracts *three* roots from any degree-three equation. Lacking a distinct third root, our equation is what’s known technically as a *degenerate case*.

1

One-Line

Theorem. *Let x be real. If $x^3 - 6x^2 + 11x - 6 = 2x - 2$, then $x = 1$ or $x = 4$.*

Proof. By subtraction, $x^3 - 6x^2 + 9x - 4 = 0$, which factors as $(x-1)^2(x-4) = 0$. $\square$

One-Line

Like poets, mathematicians often strive for economy, and the one-line proof is a sort of monostich. Even the abbreviation QED for *quod erat demonstrandum* ("which was to be shown"), which traditionally marks the end of a proof, is deemed too prolix by modern standards. Instead we find the tombstone □, sometimes called the halmos, after the Hungarian American mathematician Paul Halmos who first incorporated it into mathematical writing. Economy is an ideal that extends from proofs to larger scale works. Once, in *The Bulletin of the American Mathematical Society*, there appeared a research article written by two number theorists that, in its entirety, consisted of only two sentences.[13] I guess the authors couldn't agree on one.

In its cryptic way, the one line here does at least give the reader something to do. Go ahead, combine like terms with like—and look, divide out these common factors.

Hypothesis: $x^3-6x^2+11x-6=2x-2$, where x is a real number.
To prove: $x=1$ or 4.

2

Two-Column

	STATEMENT	REASON
1.	$x^3-6x^2+11x-6=2x-2$	Given.
2.	$x^3-6x^2+11x-6+2=2x-2+2$	Addition property of equations.
3.	$x^3-6x^2+11x-4=2x$	Addition.
4.	$x^3-6x^2+11x-4-2x=2x-2x$	Subtraction property of equations.
5.	$x^3-6x^2+9x-4=0$	Subtraction.
6.	$x^3-(1+5)x^2+(5+4)x-4=0$	Addition.
7.	$x^3-x^2-5x^2+5x+4x-4=0$	Distributive property.
8.	$x^2(x-1)-5x(x-1)+4(x-1)=0$	Factoring.
9.	$(x^2-5x+4)(x-1)=0$	Factoring.
10.	$[x^2-(1+4)x+4](x-1)=0$	Addition.
11.	$(x^2-x-4x+4)(x-1)=0$	Distributive property.
12.	$[x(x-1)-4(x-1)](x-1)=0$	Factoring.
13.	$[(x-4)(x-1)](x-1)=0$	Factoring.
14.	$x-1=0$ or $x-4=0$	Zero product property.
15.	$x-1+1=1$ or $x-4+4=4$	Addition property of equations.
16.	$x=1$ or $x=4$	Addition.
		QED

The form is familiar to American high school geometry students, who may not be wrong in guessing that it was devised for ease of grading rather than as a model of understanding. The addition of fifteen lines over the last proof does not occasion a corresponding addition in insight, no matter how much reassurance we get from that vertical line separating the action of the proof on the left from its justification on the right.

Moreover, the gains in logical transparency come at a rhetorical cost. The two-column method absolves the student from having to bother with style, not to mention grammar. According to Patricio Herbst, Professor of Educational Studies and Mathematics at the University of Michigan, this may have been one of the intentions of the two-column proof. In "Establishing a Custom of Proving in American School Geometry: Evolution of the Two-Column Proof in the Early Twentieth Century," he writes,

> as students had thus far been used to memorizing the demonstrations of a geometry text, the mental discipline that geometry made possible was being lost. Instructional changes were needed in order to enable geometry to do its job. . . .[The two-column format] gave students an "objective" representation that enabled smoother recognition of the similarities between such different activities as proving fundamental propositions and solving proof exercises.[14]

See 18 Indented for an elaboration on the two-column proof.

3

Illustrated

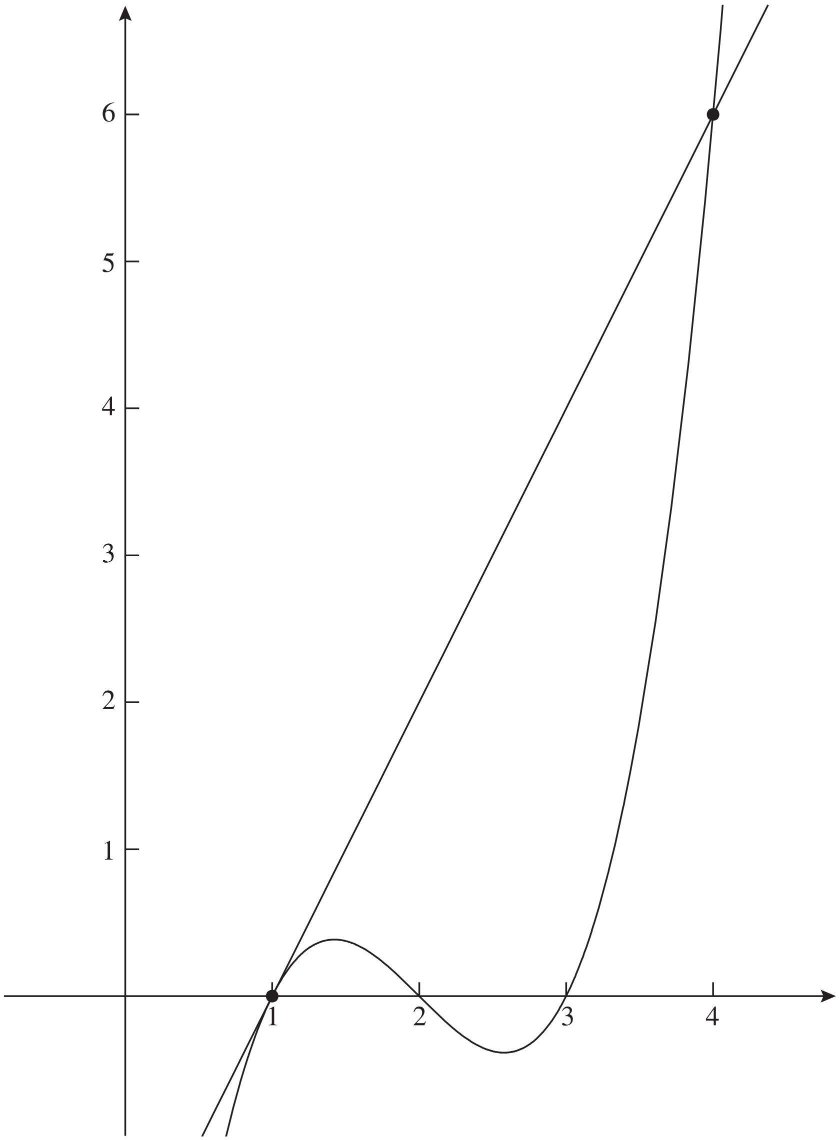


FIGURE. The two points of intersection of the cubic $y = x^3 - 6x^2 + 11x - 6$ and the line $y = 2x - 2$ occur at $(1, 0)$ and $(4, 6)$.

When I showed a draft of a few dozen proofs to a physicist colleague, he declared this one *the* proof. In a sense he was right. I originally conceived of the proposition as a claim about the intersection of two graphs.

But this example of "proof by inspection" wouldn't qualify as proof according to most mathematicians' standards. How can we be sure that the curves really do meet under those two beauty spots? What's going on outside the region plotted? I'm reminded of a story told by the French geometer Étienne Ghys. After giving a talk in the Bourbaki seminar (see the comments in 6 Axiomatic) on a geometric construction that included illustrations, he was approached by the eminent French mathematician and Fields Medalist Jean-Pierre Serre, who remarked, "That was interesting what you said. I have a question. Would you consider this to be a theorem?"[15] Nevertheless, computer plots like this as well as 12 Ruler and Compass diagrams and 21 Blackboard sketches are effective tools for discovering and communicating mathematics.

4

Elementary

Proposition. *If x is a real number and $x^3 - 6x^2 + 11x - 6 = 2x - 2$, then $x = 1$ or $x = 4$.*

Proof. The degree three equation admits the standard form $x^3 - 6x^2 + 9x - 4 = 0$. Expand the linear term $9x$ as the sum $5x + 4x$ so that a common factor is apparent on the left hand side.

$$(x^3 - 6x^2 + 5x) + (4x - 4) = (x^2 - 5x)(x - 1) + 4(x - 1)$$

Factoring out $x - 1$ leaves a quadratic, which also factors easily.

$$(x^2 - 5x + 4)(x - 1) = (x - 4)(x - 1)(x - 1)$$

Since the right side of the equation is zero, one of the factors $(x - 1)$ or $(x - 4)$ must be zero. Thus $x = 1$ or $x = 4$.

Elementary

Economy of means can be as valuable as economy of form. I understand a proof to be *elementary* if its argument only relies on techniques that are basic in the field to which the proposition belongs, judging from its terminology. In this sense, an elementary proof is a naturally occurring constraint for mathematical *reasoning*. Other forms of reasoning or methods include 13 Reductio ad Absurdum and 22 Substitution.

Authors of textbooks and mathematics curricula often try to organize their material so that the student only sees problems that are elementary in this sense. This is likely responsible for the widely held assumption that mathematics itself develops according to the same logical pattern. An example to the contrary is the prime number theorem, which describes how the average gap between primes grows as primes increase. This result belongs to the field of number theory, but it took half a century for mathematicians to find a number theoretic proof for the theorem, which was first proved by methods using complex analysis.

Is there an elementary style of writing mathematics? If so, I think it's best captured by the eminent Hungarian mathematician and educator Georg Pólya:

> Rules of style: The first rule of style is to have something to say. The second rule of style is to control yourself when, by chance, you have two things to say; say first one, then the other, not both at the same time.[16]

Suppose that among four consecutive numbers, the product of the first three equals twice the third. What's the fourth number?

5

Puzzle

Answer: 4

Maybe the first question is, "Why bother with an equation?" There are worse things to worry about. In an article entitled "What is Mathematics For?", American mathematician Underwood Dudley concludes: "In mathematics problems can be solved, using reason, and the solutions can be checked and shown to be correct.... That is what mathematics education is for and what it has always been for: to teach reasoning, usually through the medium of silly problems."[17]

The 68 Word Problem or story problem is probably the most recognizable genre of silly problems in mathematics education.

To see how the answer to this brain-teaser solves our equation (or doesn't), call the fourth number in question x. Then the three consecutive numbers prior to x (inferring that they are integers) are, in numerical order, $x-3$, $x-2$, $x-1$. If their product equals twice the third, then $(x-3)(x-2)(x-1)=2(x-1)$. Multiply these numbers and the result is a big mess, but a mess that can be tidied into the form of the equation in 0 Omitted.

6

Axiomatic

Notations

Zero and *one* are numbers, and they are denoted by 0 and 1 respectively. The *sum* of numbers x and y is the result of adding x and y, and it is denoted by $x+y$; their *product* is the result of multiplying x and y, and it is denoted by $x \times y$ or xy. Numbers x and y are *equal* if they are identical, a fact which is denoted by the equation $x=y$.

Definitions

1. The numbers 2 through 11 are defined by the sums $2=1+1$, $3=2+1, \ldots,$ $11=10+1$.
2. The *additive inverse* of a number x is the number $-x$ such that $x+(-x)=0$.
3. The *difference* of two numbers x and y is denoted $x-y$, and it is defined as the sum $x+(-y)$.
4. The *square* of a number x is the the product of x with itself, and it is denoted by x^2.
5. The *cube* of a number x is the product of x with its square, and it is denoted by x^3.

Axioms

6. Given any proposition P, if P or P, then P.
7. For all numbers x and y, if $x=y$, then $y=x$.
8. For all numbers x, y, z, if x is equal to y and y is equal to z, then x is equal to z.
9. For all numbers x and y, and any equation E, if x is equal to y, then y may be substituted for any occurrence of x in E without changing the truth value of E.
10. If x and y are numbers, then the sum $x+y$ and product $x \times y$ are numbers also.
11. For all numbers x, y, z, if x is equal to y, then the sums $x+z$ and $y+z$ are equal, as are the products $x \times z$ and $y \times z$.
12. For all numbers x, y, the transposed sums $x+y$ and $y+x$ are equal, as are the transposed products $x \times y$ and $y \times x$.
13. For all numbers x, y, z, the triple sums $(x+y)+z$ and $x+(y+z)$ are equal, as are the triple products $(x \times y) \times z$ and $x \times (y \times z)$.
14. If x, y, z are numbers, then the product of x with the sum $y+z$ is equal to the sum of the products $x \times y + x \times z$.
15. The number 1 is not equal to the number 0.
16. For any number x, the sum $0+x$ is equal to x.
17. For any number x, the product $1 \times x$ is equal to x.
18. For any number x, there exists a unique additive inverse $-x$.
19. For any numbers x, y, if $x \times y = 0$, then $x=0$ or $y=0$.

Theorems

20. For all numbers x, y, z, if $x = y$, then $x - z = y - z$.
21. For any number x, $x - x = 0$.
22. For any number x, $0 \times x = 0$.
23. For any numbers x, y, $(-x)y = -(xy) = x(-y)$
24. For any number x, $-(-x) = x$
25. For any numbers x, y, z, $x(y - z) = xy - xz = (y - z)x$
26. For any numbers x, y, z, w, $(x - y)(z - w) = xz - xw - yz + yw$
27. For any number x, $x + x = 2x$
28. For any numbers x, y, $-(x + y) = -x - y$
29. $-2 + (-4) = -6$
30. $1 + 4 \times 2 = 9$
31. For any number x, $(x - 1)^2 = x^2 - 2x + 1$.
32. For any number x, $(x - 1)^2(x - 4) = x^3 - 6x^2 + 9x - 4$.
33. For any number x, $x^3 - 6x^2 + 9x - 4 = (x^3 - 6x^2 + 11x - 6) - (2x - 2)$.
34. For any number x, if $x^3 - 6x^2 + 11x - 6 = 2x - 2$, then $x = 1$ or $x = 4$.

PROOF. Suppose x is a number.

Theorem 33	$x^3 - 6x^2 + 9x - 4 = (x^3 - 6x^2 + 11x - 6) - (2x - 2)$	(1)
Hypothesis	$x^3 - 6x^2 + 11x - 6 = 2x - 2$	(2)
Axiom 10	$2x - 2$ is a number	(3)
Axiom 9, (1), (2), (3)	$x^3 - 6x^2 + 9x - 4 = (2x - 2) - (2x - 2)$	(4)
Theorem 21, (3)	$(2x - 2) - (2x - 2) = 0$	(5)
Axiom 8, (4), (5)	$x^3 - 6x^2 + 9x - 4 = 0$	(6)
Theorem 32	$(x - 1)^2(x - 4) = x^3 - 6x^2 + 9x - 4$	(7)
Axiom 8, (7), (6)	$(x - 1)^2(x - 4) = 0$	(8)
Axiom 19, (8)	$(x - 1)^2 = 0$ or $x - 4 = 0$	(9)
Definition 4, (9)	$(x - 1)(x - 1) = 0$ or $x - 4 = 0$	(10)
Axiom 19, (10)	$x - 1 = 0$ or $x - 1 = 0$ or $x - 4 = 0$	(11)
Axiom 6, (11)	$x - 1 = 0$ or $x - 4 = 0$	(12)
Definition 3, (12)	$x + (-1) = 0$ or $x + (-4) = 0$	(13)
Axiom 11, (13)	$x + (-1) + 1 = 0 + 1$ or $x + (-4) + 4 = 0 + 4$	(14)
Definition 2, (14)	$x + 0 = 0 + 1$ or $x + 0 = 0 + 4$	(15)
Axiom 16, (15)	$x = 1$ or $x = 4$	(Theorem)

The influential German mathematician David Hilbert offers one of the most succinct descriptions of the axiomatic approach: "If we consider a particular theory more closely, we always see that a few distinguished propositions of the field of knowledge underlie the construction of the framework of concepts, and these propositions then suffice by themselves for the construction, in accordance with logical principles, of the entire framework."[18] This proof is based on Italian mathematician Giuseppe Peano's *The principles of arithmetic, presented by a new method* from 1889.[19] The theorem is proved at the end of a sequence of theorems, each of which relies on one or more of the axioms, definitions, and theorems that precede it. Terms that are deemed too primitive to warrant precise definition are considered mere notations. It may seem absurd to consider a simple equation like $1+4\cdot 2=9$ to be a theorem, but it is a result that can be logically deduced from the axioms stated.[20]

The axiomatic method of organizing knowledge into a logical hierarchy dates back at least as far as Euclid (see 52 Antiquity), but axiomatic systems took on a new character and prominence in the modern era. The group of young mathematicians writing collectively under the pseudonym Nicolas Bourbaki sought to organize vast tracts of the discipline along the lines of the modern axiomatic style inspired by Hilbert and the celebrated algebraist Emmy Noether.[21] Bourbaki's 1948 manifesto, "The Architecture of Mathematics," explains their perspective (emphasis added):

> From the axiomatic point of view, mathematics appears thus as a storehouse of abstract forms—the mathematical structures.... Of course, it can not be denied that most of these forms had originally a very definite intuitive content; but, it is exactly by *deliberately throwing out this content*, that it has been possible to give these forms all the power which they were capable of displaying and to prepare them for new interpretations.[22]

Setting aside the brazenness of these lines, it's not difficult to imagine that some might have a hard time going along with such a formalist program. Without questioning the benefits afforded by the Bourbaki approach, mathematician and philosopher Gian-Carlo Rota observed at the end of the century:

> The axiomatic method of presentation of mathematics has reached a pinnacle of fanaticism in our time.... Clarity has been sacrificed to such shibboleths as consistency of notation, brevity of argument and the contrived linearity of inferential reasoning. Some mathematicians will go as far as to pretend that mathematics *is* the axiomatic method, neither more nor less. This pretense of "identifying" mathematics with a style of exposition is having a corrosive effect on the way mathematics is viewed by scientists in other disciplines.[23]

Other mathematicians will find that mathematics itself pays a price for our over reliance on formalism; see the comments following 33 Calculus.

linquitur 1, cuius R cubicam quæ est 1, detrahe ex 2, τp̄q̄d. relinquitur 1, rei æstimatio.

Quòd si numerus positionum, maior sit producto ex numero quadratorum in sui partem tertiam, differentia erit numerus rerum, ut in prima demonstratione, & suis regulis, hunc duc in τp̄q̄d. & ei adde cubum τp̄q̄d. & huius aggregati, numeriq; æquationis differentia, est numerus æquationis cubi, & talium rerum differentia, si nulla sit, æstimatio rei est τp̄q̄d. Et si numerus æquationis est minor aggregato, æstimationem inuentam minue, & si maior, adde τp̄q̄d. quod fiet, erit rei æstimatio. Exemplum, cubus & 20 res, æquantur 6 quadratis & 24, ducto 6 in 2, tertiam partē sui, fit 12, cuius differentia à 20, numero rerum, est 8, numerus rerum, quæ cum cubo æquantur numero, duc igitur 8 numerum rerum, in 2 τp̄q̄d. fit 16, adde ei 8, cubum τp̄q̄d. fit 24, differentia cuius nulla est à 24 numero æquationis, igitur æstimatio rei est τp̄q̄d. scilicet 2, sit rursus cubus cū 20 rebus, æqualis 6 quadratis & 15, habebimus igitur, ut prius, cubum & 8 res, pro numero, duc ut prius, 8 numerum rerum posteriorem in 2 τp̄q̄d. fit 16, adde cubum τp̄q̄d. fit 24, abijce 15, relinquitur 9, igitur cubus & 8 res, æquatur 9, & rei ęstimatio est 1, quod minue ex 2, τp̄q̄d. relinquitur uera æstimatio rei 1, minuisti autem, quia 15 numerus æquationis, est minor aggregato cubi & producti, quod est 24, & si bene animaduertis, eodem modo fit in prima parte regulæ, quando numerus rerum æqualis est producto ex numero quadratorum in sui partem tertiam. Rursus, cubus cum 20 rebus, æqualis sit 6 quadratis p: 33, habebis itaq; cubum, ut prius, & 8 res, æquales differentiæ 24 aggregati, & 33 numeri æquationis, quare cubus & 8 res, æquabuntur 9, & æstimatio rei erit 1, addendum τp̄q̄d. quia numerus æquationis 33, est maior numero aggregato 24, quare rei æstimatio erit 3.

Quòd si numerus positionum, minor sit producto ex numero quadratorum in sui tertiam partem, differentia nihilominus erit numerus rerum, ut prius, sed hæ non copulabuntur cubo, imò erunt ei æquales, deinde duc ipsum numerum rerum posteriorum, in τp̄q̄d, & productū iunge numero æquatiōis, huius aggregati & cubi τp̄q̄d. differentia est numerus æquationis secundæ, si igitur differentia nulla est, cubus æquabitur rebus, & R quadrata numeri rerum addita τp̄q̄d. est æstimatio rei, quod si aggregatum sit maius cubo, erit differentia, numerus qui cū rebus ęquatur cubo, inde habita æstimatione, adde ei τp̄q̄d. & fiet uera æstimatio. Quòd si cubus fuerit maior aggrega-

gregato, differentia erit numerus, qui cum cubo æquatur rebus, inde habita æstimatione, adde ei ℞ p̃q̃d. quod conflatur, est rei uera æstimatio, & tam multiplex habenda, ut in nostra regula docuimus, quanquam quod ad regulam pertinet, & hæc nostra sit. Exemplũ igitur, Cubus & 9 res, æquales sint 6 quadratis p: 2, tunc numerus rerum secundus erit 3, duc in 2, ℞p̃q̃d. fit 6, adde ad 2 numerum æquationis, fit 8, cubus autem ℞p̃q̃d. est 8, differentia nulla, igitur cubus æquatur 3 rebus, res igitur est ℞ 3, & rei æstimatio 2 p: ℞ 3. Rursus, cubus p: 9 rebus, æqualis sit 6 quadratis p: 4, habebimus ut prius, cubum æq̃lem 3 rebus, pro numero duc 3 numerum rerum posteriorem in 2 ℞p̃q̃d. fit 6, adde 4, numerum equationis, fit 10, abijce 8, cubum ℞p̃q̃d. fit 2, addendus rebus, quia aggregatum est maius cubo ℞p̃q̃d. igitur cubus æquatur 3 rebus, p: 2, & res erit 2, addito 2 ℞p̃q̃d. fit 4, uera æstimatio. Iterum, sit cubus p: 21 rebus, æqualis 9 quadratis p: 5, erunt igitur 6 res in posteriore æquatione, quia 9 numerus quadratorum, ductus in 3, tertiam sui partem, producit 27, duc igitur 6 numerum posteriorem rerum, in 3, ℞p̃q̃d. fit 18, adde ei 5, fit 23, differentia cuius à numero producto ex cubo c ℞p̃q̃d. est 4, & quia aggregatũ est minus cubo, ideo cubus & 4, æquabuntur 6 rebus, æstimatio igitur est 2, uel ℞ 3 m: 1, & ficta ℞ 3 p: 1, quæ est m: si igitur his addas 3 ℞p̃q̃d. habebis æstimationes quæsitas 5, & 4 p: ℞ 3, & 2 p: ℞ 3, in harum qualibet uerum est, quod cubus & 21 res, æquales sunt 9 quadratis & 5 numero.

De cubo & quadratis, æqualibus rebus & numero. Caput XIX.

DEMONSTRATIO.

Sit etiam cubus A B, & 6 quadrata, æq̃lia 20 rebus p: 200, gratia exempli, & ponemus B C 2, ℞p̃q̃d. erit igitur A C res p: 2, & eius cubus, erit cubus & 6 quadrata, & 12 res, & 8 iam autem suppositum est, quod cubus A B & 6 quadrata, sint æqualia 20 rebus p: 200, igitur ponantur, 20 res & 200, loco cubi, & 6 quadratorum, & fiet cubus A C, æqualis 32 rebus p: 208, at quia 32 res A B, deficiũt à 32 rebus A C, in 32 B C, addantur utrique parti 32 B C, erunt igitur 32 res p: 208, æquales cubo p: 64, tantum enim sunt 32 B C, abijce 64 ab utraque parte, erit cubus æqualis 32 rebus p: 144, inde inuenta æstimatione abijce B C, ℞p̃q̃d. relinquetur A B.

REGVLA.

I was stunned to find my humble cubic in the middle of *Ars Magna*, the 1545 treatise on cubic equations by Italian Renaissance physician, astrologer, scientist, and mathematician Girolamo Cardano.[24] But in fact *most* cubics with small positive integer roots are likely to appear in one of Cardano's forty chapters. Why so many chapters? Prior to the incorporation of negative numbers into algebra, two equations like $ax^3+bx^2+cx=d$ and $ax^3+bx^2=cx+d$ would fall into separate *categories* of cubic equation, and Cardano treats each separately and with several examples.

Like other Renaissance mathematicians, Cardano wrote in an awkward, proto-algebraic shorthand that included p: and m: for addition and subtraction and Tpq̃d to express one-third the coefficient of the square (*tertia pars numeri quadratorum*). General rules are introduced with perfunctory geometric arguments before proceeding to computational examples such as this.

Using modern notation, Richard Witmer translates the underlined passage as follows:

> Again,
>
> $$x^3+9x=6x^2+4.$$
>
> We shall, as before, have y^3 equal to $3y$. For the number, multiply 3, the coefficient of y, by 2, one-third the coefficient of x^2, making 6. Add 4, the constant of the equation, making 10. Subtract 8, the cube of one-third the coefficient of x^2, and 2 is the result. This is to be added to the y's, because the sum is greater than the cube of one-third the coefficient of x^2. Hence
>
> $$y^3=3y+2,$$
>
> and y will be 2. By adding 2, one-third the coefficient of x^2, 4 results as the true solution [for x].[25]

For a discussion of Cardano's method, including the change of variable from x to y, see 25 Open Collaborative.

Ars Magna presented the first comprehensive solution of the cubic in print, and for that reason the modern cubic formula usually bears his name (see 30 Formulaic). Despite his acknowledgment of the contributions of Scipione del Ferro and "my friend Niccolò Tartaglia" in the first chapter, the book sparked a fierce and well-documented priority battle (see 43 Screenplay).[26]

Why does Cardano designate 4 as the *vera aestimatio*? What's untrue about 1?

8

Prerequisite

Theorem. *Let* $x \in \mathbb{R}$. *If* $x^3 - 6x^2 + 11x - 6 = 2x - 2$, *then* $x = 1$ *or* $x = 4$.

Proof. Artin [Ch. 14, §2], or Herstein [§5.7].

Unlike 7 Found, neither of the two sources cited here, *Algebra* by Michael Artin or *Topics in Algebra* by I. Herstein, solve our cubic *per se*.[27] Instead they discuss the solution to any and all cubic equations no matter what their coefficients.

In general, it's not clear what it means to say that two proofs are "the same." It may, however, be easier to provide criteria for two proofs to differ. The English mathematician John Conway, famous for his gaming approach to mathematics, and Joseph Shipman, a mathematical consultant to casinos, suggest that proofs can be partially ordered by pitting them against one another in competition. They find that every proof has a "natural domain of applicability" or "scope," and one proof bests another if its scope contains the scope of the other.[28] It seems reasonable that if one proof extends, *mutatis mutandis*, to a case beyond the reach of a second proof, then they must be different. For example, Cardano uses the same method to solve both our cubic and $x^3+21x=9x^2+5$ (it appears just below the underlined text in 7 Found). The solutions of this second cubic are $2-\sqrt{3}$, $2+\sqrt{3}$, and 5. These roots are definitely out of reach of the 6 Axiomatic proof, since that proof begins by assuming x is integral whereas $2-\sqrt{3}$ and $2+\sqrt{3}$ aren't even rational numbers.

Conway and Shipman are quick to point out that the value we place on a proof need not correlate to the dominance of its scope. For example, a proof with a huge scope relative to what it actually proves, like "swatting flies with a sledgehammer," is usually deemed excessive and graceless.[29] 64 Research Seminar provides one such example.

9

Monosyllabic

Here is a fact:

If x is real and the cube of x less six times the square of x plus five times x plus six times x less six is twice x less two, then x must be one or four.

The proof goes like this:

See, the first three terms on the left side split as the square of x less five times x all times x less one. And more, the last two terms on the left side split as six times x less one, while the right side splits as two times x less one. Thus, if x were to be one, we have nought plus nought is nought, which is true. So, x may be one.

Else x is not one, and x less one is not nought. So we can times the whole thing by one on top of x less one to yield: the square of x less five times x plus six is two. Drop two from each side, and the square of x less five times x plus four is nought. Now this splits as x less four times x less one. Since we said x less one is not nought, x less four must be. So x is one or four, as was to be shown.

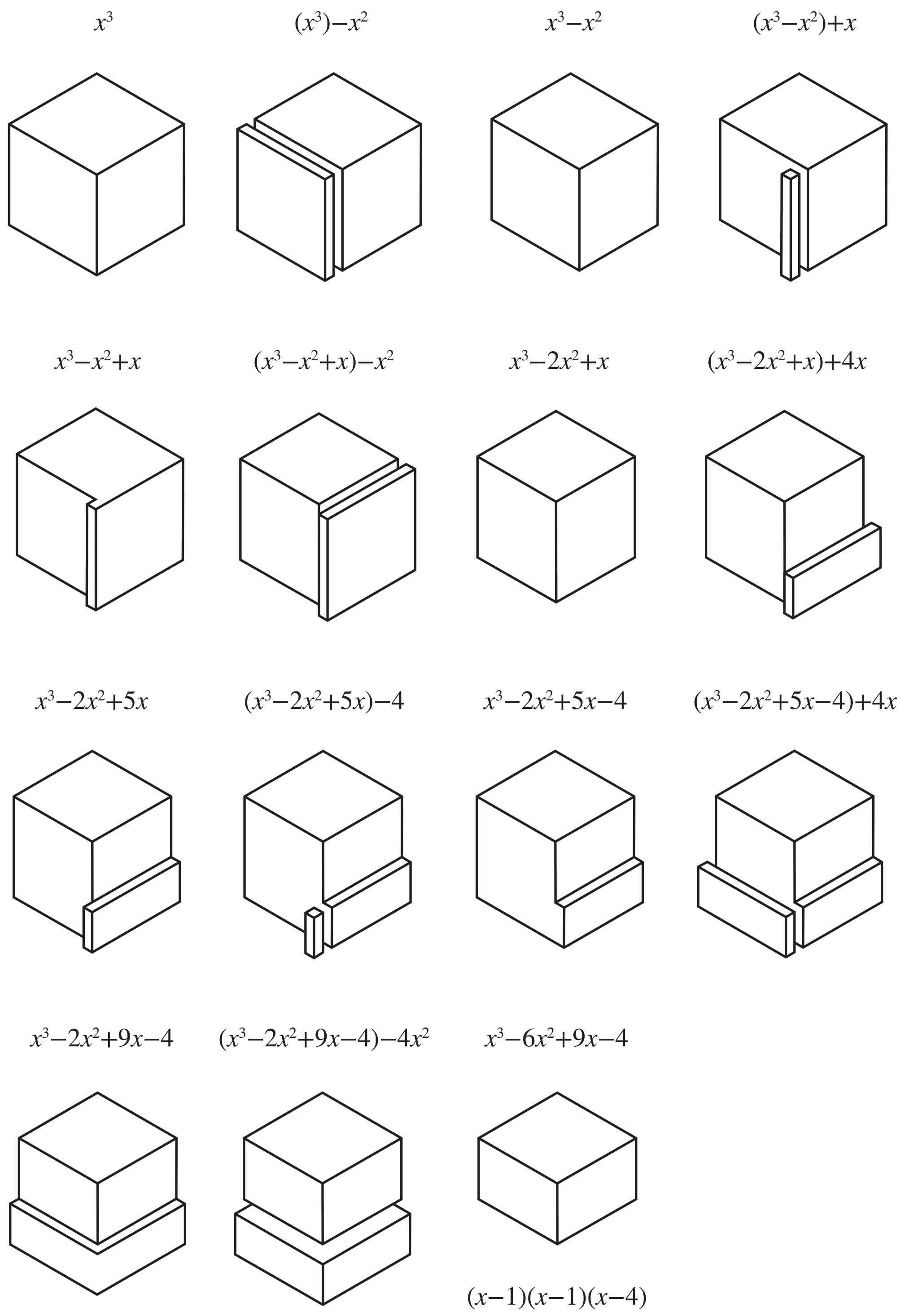
In Siobhan Roberts' biography *Genius at Play: The Curious Mind of John Horton Conway*, the distinguished mathematician claims that he delivered an entire number theory lecture according to the rules of the "One Bit Word Game," which is an autological title for the monosyllabic constraint.[30]

In place of *eleven*—the only polysyllabic coefficient in the equation—I've written "five plus six." This trivial alteration led to the factorization that is the basis of the solution here (and in 4 Elementary). Why am I so surprised by this? Mathematical proofs often emerge as a sequence of individually inconsequential transformations governed by external constraints. It could even be that this is typical.

In the 1948 article, "The Place of Mathematics in the Classification of the Sciences," Raymond Queneau argues that mathematics is both method and game, "in the most precise terms, what's known as a *jeu d'esprit*." He concludes the paper by signaling a proto-Oulipian correspondence: "We might say, giving Art its ambiguous sense, that Science oscillates from Art to Game and Art from Game to Science."[31]

10

Wordless



x^3
$(x^3)-x^2$
x^3-x^2
$(x^3-x^2)+x$
x^3-x^2+x
$(x^3-x^2+x)-x^2$
x^3-2x^2+x
$(x^3-2x^2+x)+4x$
x^3-2x^2+5x
$(x^3-2x^2+5x)-4$
x^3-2x^2+5x-4
$(x^3-2x^2+5x-4)+4x$
x^3-2x^2+9x-4
$(x^3-2x^2+9x-4)-4x^2$
x^3-6x^2+9x-4
$(x-1)(x-1)(x-4)$

Wordless

Some diagrams or illustrations are so compelling that they seem to hold up on their own without explanation. Also known as "Look-see! Proofs" or Proofs Without Words (PWWs), these have been a feature of mathematics magazines ever since they first appeared in the 1970s.[32]

The question of their legitimacy as proofs seems beside the point, but every discussion of PWWs raises such disclaimers. Writing for the Mathematical Association of America's online journal *Convergence*, Tim Doyle et al. frame the debate in stylistic terms:

> We use the term "Baroque" to indicate the idea that status as a mathematical proof is underwritten by formal correctness, paralleling how a composition's status as a fugue is underwritten by the formal qualities of the music. . . . We use the term "Romantic" to indicate that a good proof is whatever sparks the mathematical intuition or intellect in the right way, prioritizing the mathematical experience over Baroque genre standards. The Baroque perspective construes proof as a type of evidence distinguished by meeting particular formal constraints, whereas the Romantic perspective construes any evidence that is powerfully and completely compelling to a mathematically astute, reasonable reader as a proof.

This is a useful framework, not least because it allows the authors to refine their position in the debate, hedging their bets with the precision of an analyst:

> We have to be careful not to take the Baroque perspective so far that it turns out nobody ever writes proofs! When we relax the standard a little bit it becomes interesting to ask whether less canonical forms of proof write-up, like PWWs, can slip through the gap and have a place within—or perhaps just in the neighborhood of—the category of proofs.[33]

This proof resulted from work on 52 Antiquity. I've drawn the starting cube x^3 with dimensions $10 \times 10 \times 10$ units. What happens to the proof as the side of the cube approaches a solution, $x = 1$ or $x = 4$? Can you see the proof in fewer steps? 62 Axonometric offers a one-step wordless proof.

Instructions: Write your answer to the question directly on this page. All work should be written in blue or black pen. Clearly show all appropriate algebraic work. Scrap paper is not permitted, but you may use the reverse side of this page for scratch work. The use of any communications device is strictly prohibited when taking this examination. If you use any communications device, no matter how briefly, your examination will be invalidated, and no score will be calculated for you.

11
Exam

Solve $x^3 - 6x^2 + 11x - 6 = 2x - 2.$

$$x^3 - 6x^2 + 11x - 6 = 2x - 2$$
$$-2x + 2 \quad -2x + 2$$

$$x^3 - 6x^2 + 9x - 4 = 0$$
$$\cancel{x^2(x-6) + 9(x-4) = 0}$$

$$x^3 - 6x^2 + 11x - 6 = \frac{2(x-1)}{(x-1)}$$

$$
\begin{array}{r}
x^2 - 5x + 6 \\
x - 1 \overline{)\, x^3 - 6x^2 + 11x - 6} \\
-x^3 + x^2 \\
\hline
-5x^2 + 11x \\
+5x^2 - 5x \\
\hline
6x - 6 \\
-6x + 6 \\
\hline
0
\end{array}
$$

$$x^2 - 5x + 6 = 2$$
$$\cancel{(x-2)(x-3) = 2}$$
$$x^2 - 5x + 4 = 0$$
$$(x-1)(x-4) = 0$$

$x = 1$
$x = 4$

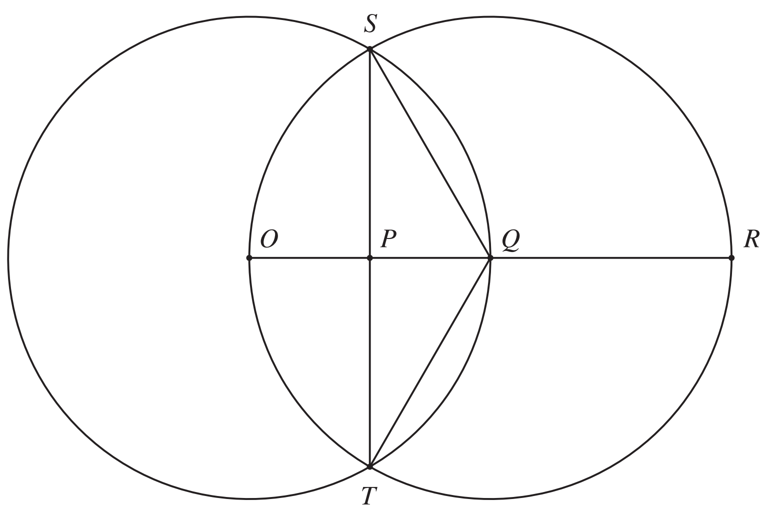
Exam

The most vivid form of mathematics for many is likely to be the exam. “Math test” is a powerful linguistic collocation not unlike “splitting headache” or “excruciating pain.” This variation, including its threatening Instructions, is modeled on the New York State Algebra 2 / Trigonometry Regents Examinations.[34]

In fact, most standardized mathematics tests from elementary school to the GRE ask multiple-choice questions for which no evidence of reasoning is required on the part of the student. When such tests do assess reasoning, it’s interesting that standards may vary depending on the subject matter. In the case of the Regents Examinations, the grading rubric instructs graders to assign full credit when the correct numerical solution and “appropriate algebraic work is shown,” but for geometry “a complete and correct proof that includes a concluding statement” is required. Why? Perhaps the higher standard of proof for geometry reflects the long legacy of Euclid’s *Elements* and its 6 Axiomatic style.

The roots of the cubic polynomial $x^3 - 6x^2 + 11x - 6 - (2x - 2)$, or $x^3 - 6x^2 + 9x - 4$, may be constructed as follows:

1. Place two points O and P arbitrarily.
2. Construct the multiples of length OP for the given coefficients, $b = -6$, $c = 9$, and $d = -4$.
3. Construct lengths for each of the following two auxiliary coefficients
$$p = \frac{b^2 - 3c}{9} = 1 \qquad q = \frac{2b^3 - 9bc + 27d}{54} = -1.$$
4. Construct a point Q on the extended line OP such that the length OQ is $-b/3 = 2$.
5. Draw the circle OSR centered at Q with radius $2\sqrt{p} = 2$.
6. Draw the circle QST with the same radius centered at O.
7. Draw the lines PS and PT. Each of these lines are perpendicular to OQ.
8. The double root is the length OP and the single root is the length OR.



Ruler and Compass

These tools—the unmarked ruler (or straightedge) and compass—have given mathematicians more to do and think about than would seem reasonable, considering their modesty. The most basic and far-reaching question that they provoke is, simply, what *can* and *cannot* be measured?

One of the classical construction problems is to trisect an arbitrary angle (41 Newsprint cites another, doubling the cube). The angle trisection problem happens to be equivalent to the problem of finding roots of an arbitrary cubic polynomial x^3+bx^2+cx+d. Unfortunately, both angle trisections and cube roots are, in general, *not* constructible by straightedge and compass. If, however, the cubic has repeated real roots, as our cube does, then its solutions are constructible. (If a cubic has three real roots, then a *marked* straightedge and compass will find the roots.[35]) The auxiliary coefficients p, q correspond to a particular form of the reduced cubic, $x^3-3px+2q$, and a cubic polynomial will have repeated roots if and only if $p^3=q^2$.

To construct the roots, we apply a 47 Clever trigonometric identity:

$$4\cos^3\theta-3\cos\theta=\cos 3\theta.$$

The above cubic equation in $\cos\theta$ is perfectly adapted to our given cubic when $x=2\cos\theta+2$ and $\cos 3\theta=1$. In coordinate geometry, the first of these two conditions means that a root x is the horizontal coordinate of the point θ on the circle of radius 2 centered at $Q=(2,0)$. The second condition implies that θ is one-third of a multiple of 2π; that is, θ equals 0, $2\pi/3$, or $4\pi/3$. These angles correspond to the points R, S, and T.

13

Reductio ad Absurdum

Theorem. *Let x be a real number. If $x^3-6x^2+11x-6=2x-2$, then $x=1$ or $x=4$.*

Proof. That $x=1$ and $x=4$ are solutions is readily verified. Suppose there is a third solution x such that $x\neq 1$ and $x\neq 4$. Then we may divide $x^3-6x^2+11x-6=2x-2$ by $x-1$ to obtain $x^2-5x+6=2$, that is, $x^2-5x+4=0$. Now dividing by $x-4$ leaves $x-1=0$. Dividing again by $x-1$ we conclude $1=0$, which is absurd. Hence $x=1$ or $x=4$ as required. $\square$

Reductio ad Absurdum

Reductio ad absurdum is an indirect method of proof, similar to and often confused with 14 Contrapositive. While both methods begin by supposing that the desired conclusion is false, the aim of a proof by contradiction is not to deduce that the hypothesis is false (which isn't contradictory), but to show that some third statement that was already known to be true—an axiom or proven proposition—is false. In this case, the contradiction is that 1 equals 0.

What counts as absurd depends on your logic, and some logicians take issue with the *reductio*. One concern is the assumption that rejecting the negation of a proposition is tantamount to proving it is true. Twentieth century Dutch mathematician and philosopher L. E. J. Brouwer was the most famous critic of this assumption, known as the law of excluded middle. This isn't to say that he was an advocate of a third truth value (I think that would actually be an example of *tertium-non-datur* reasoning). See 79 Intuitionist. Another concern is that when we accept proofs by contradiction, we tacitly assume that our rules of proof are consistent (i.e., that one cannot prove *A* and not *A*). To see what happens otherwise, go to 82 Inconsistency.

Even if you're willing to exclude the middle and take consistency on faith, extracting meaning from an indirect proof is complicated by the distortion wrought upon the proposition in the proving. This can be a rhetorical strength, however: Pólya notes "'Reduction to an absurdity' is a mathematical procedure but it has some resemblance to irony which is the favorite procedure of the satirist. Irony adopts, to all appearance, a certain opinion and stresses it and overstresses it till it leads to a manifest absurdity."[36]

14

Contrapositive

Theorem. *Let x be a real number. If $x^3 - 6x^2 + 11x - 6 = 2x - 2$, then $x = 1$ or $x = 4$.*

Proof. Suppose $x \neq 1$ and $x \neq 4$. Then $(x-1)(x-1)(x-4) \neq 0$, since the factors $(x-1)$ and $(x-4)$ are nonzero. Now

$$\begin{aligned}(x-1)(x-1)(x-4) &= x^3 - 6x^2 + 9x - 4 \\ &= (x^3 - 6x^2 + 11x - 6) - (2x - 2),\end{aligned}$$

which must also be different than zero. Hence $x^3 - 6x^2 + 11x - 6 \neq 2x - 2$, and we have proved by contraposition that the only possible roots are 1 and 4. That these are indeed roots is easily verified. □

Contrapositive

The following syllogism probably dates back to the Stoic philosophers of the third century BCE:

> If it is day, it is light.
> But it is not the case that it is light.
> Therefore it is not the case that it is day.[37]

This is a typical example of the logical equivalence between a conditional statement *if P, then Q* and its contraposition *if not Q, then not P*, also known as *modus tollens*.

The careful reader will notice that the indirect arguments, both in this exercise and 13 Reductio ad Absurdum, are supplemented by a casual remark about verifying that 1 and 4 are roots. This is because the indirect arguments deduce that 1 and 4 are the only *possible* roots but leave open the question of whether or not the equation even admits a solution in the first place. A friend taught me not to forget about vacuous suppositions with an old math joke:

> *Teacher*: Suppose x is the number of sheep in the problem.
> *Student*: But, Sir, what if x is not the number of sheep?

15

Matrices

Theorem. *Let $\lambda \in \mathbb{R}$. If λ is an eigenvalue of the linear operator*

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

then $\lambda = 1$ or $\lambda = 4$.

Proof. A number λ is an eigenvalue of the matrix $\mathbf{A}$ if there is a non-zero vector $\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ such that

$$\mathbf{Ax} = \lambda\,\mathbf{x}.$$

This implies the homogeneous set of simultaneous equations,

$$(\mathbf{A} - \lambda\,\mathbf{I})\mathbf{x} = \mathbf{0}$$

which have a nontrivial solution when the determinant of $(\mathbf{A} - \lambda\mathbf{I})$ vanishes. Expanding the determinant of the characteristic equation gives

$$\begin{aligned} 0 &= \begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & 2-\lambda \end{vmatrix} \\ &= (3-\lambda)(1-\lambda)(2-\lambda) - 2(1-\lambda) \\ &= (1-\lambda)^2(4-\lambda). \end{aligned}$$

The roots of this equation are $\lambda_1 = 1 = \lambda_2$ and $\lambda_3 = 4$, which are therefore the eigenvalues of $\mathbf{A}$. □

This variation renders our cubic equation as the so-called characteristic polynomial associated with a three-by-three array of numbers or *matrix*.[38] Matrices originated as a bookkeeping device for solving systems of equations simultaneously, but the algebra of matrices has since established itself as a mathematical subject in its own right. Manipulating matrices is awkward at first, but they quickly become a very expressive tool. Many different mathematical objects admit matrix forms, from algebraic and differential equations to symmetries and groups (see 24 Another Symmetry for a discussion on groups).

There is a double sense to the etymology of style—style as a way of writing and an instrument of writing (thus the cognate *stylus*). The French philosopher and historian of mathematics David Rabouin underscores this twin interpretation for mathematical style:

> It is important not only to emphasize the existence of [mathematical] styles circulating across different cultural determinations but also to invest this category positively by anchoring it in the ways of writing, that is, the more material aspect of this circulation (as opposed to what is attached to "interpretations" and "meanings" shared by a community of actors or, in the opposite direction, to some platonic "ideas" grounding the supposed "universality" of the supposed "conceptual" background).[39]

In brief, "the conceptual content of a style [is] captured by the way of writing in and of itself (rather than expressed through it). . . writing is reasoning for us."

16

Ancient

𒌋𒐉	𒌋𒐉
𒌍𒈫	𒌍𒐉
𒁹 𒈫	𒁹 𒐋
𒁹 𒐐	𒁹 𒐐
𒐈 𒈫	𒈫 𒐐𒐋
𒐉 𒐐𒐉	𒐈 𒐐𒐉
𒐌 𒈫	𒐊 𒌋𒐉
𒌋 𒈫	𒐋 𒐐𒐋
𒌋𒐈 𒐐	𒐍 𒌍
𒌋𒐍 𒌍𒈫	𒌋 𒎙𒐋
𒎙𒐉 𒌋𒐉	𒌋𒈫 𒌍𒐉
𒌍𒁹 𒈫	𒌋𒐉 𒐐𒐉
𒌍𒐋 𒈫	𒌋𒐌 𒎙𒐋
𒐐𒐍 𒎙	𒎙 𒌋
𒐐𒐎 𒈫	𒎙𒐈 𒐋
𒁹 𒌋𒁹 𒌋𒐉	𒎙𒐋 𒌋𒐉
𒁹 𒎙𒐊 𒈫	𒎙𒐎 𒌍𒐉
𒁹 𒐏 𒌍𒈫	𒌍𒐈 𒐋
𒁹 𒐐𒐌 𒐐	𒌍𒐋 𒐏
𒈫 𒌋𒐌 𒈫	𒐏 𒐏𒐋
𒈫 𒌍𒐍 𒌋𒐉	𒐏𒐉 𒐐𒐉
𒐈 𒁹 𒌍𒈫	𒐏𒐎 𒌋𒐉
𒐈 𒎙𒐌 𒈫	𒐐𒐈 𒐏𒐋
𒐈 𒐐𒐉 𒐐	𒐐𒐌 𒌍

This table of numbers imagines how one of the earliest mathematical cultures on record, the Babylonians of the second millennium BCE, might express the cubic. The two columns of cuneiform list the values of each side of the equation

$$x^3 + 11x + 2 = 6x^2 + 2x + 6$$

which is the result of reorganizing $x^3 - 6x^2 + 11x - 6 = 2x - 2$ so as to avoid negative terms (without giving in to the temptation to simplify along the way). Our solutions are the row numbers—first and fourth—of those rows in which the first and second entries are equal. Those entries, 𒌋𒐘 and 𒁹 𒐐, are numerals for 14 and 110, respectively. The entire table is 17 Interpreted in Hindu-Arabic numerals in the next variation.

One of the benefits of studying the history of mathematics, according to the American mathematician Barry Mazur, is that it unifies mathematics into "*one long conversation* stretching over millennia."[40] An actual artifact from this period—the tablet known as *The Babylonian Cellar Text*—demonstrates numerous calculations of basement dimensions in terms of its length, width, or depth and the amount of excavated earth.[41] By constructing tables of values of $n^3 + n^2$, it appears that the Babylonians were able to solve equations of the form $ax^3 + bx^2 = c$.

17

Interpreted

14"	14"
32"	34"
1’2"	1’6"
1’50"	1’50"
3’2"	2’46"
4’44"	3’54"
7’2"	5’14"
10’2"	6’46"
13’50"	8’30"
18’32"	10’26"
24’14"	12’34"
31’2"	14’54"
39’2"	17’26"
48’20"	20’10"
59’2"	23’6"
1°11’14"	26’14"
1°25’2"	29’34"
1°40’32"	33’6"
1°57’50"	36’50"
2°17’2"	40’46"
2°38’14"	44’54"
3°1’32"	49’14"
3°27’2"	53’46"
3°54’50"	58’30"

Interpreted

Opening our ears to the voices captured by historical artifacts is of course not a straightforward business. This table of numbers presents an interpretation of the cuneiform in 16 Ancient. The two columns translate the Babylonian script into Hindu-Arabic numerals while preserving the base 60 positional notation, following the example of mathematical historians like Jens Høyrup of Denmark.[42] The symbols for seconds, minutes, and degrees distinguish the units, multiples of 60, and multiples of 60^2, respectively.

For a modern variation on this and the previous exercise, see 84 Tabular.

18

Indented

Theorem. $x \in \mathbb{R},\ x^3 - 6x^2 + 11x - 6 = 2x - 2 \Rightarrow x = 4 \vee x = 1.$

Proof. $x^3 - 6x^2 + 11x - 6 = 2x - 2$

$\equiv \langle \text{Subtract } 2x-2 \rangle$

> $\equiv \langle \text{Subtraction prop.} \rangle\ x^3 - 6x^2 + 11x - 6 - (2x-2) = 2x - 2 - (2x-2)$
>
> $\equiv \langle \text{Simplify} \rangle\ x^3 - 6x^2 + 9x - 4 = 0$

$\equiv \langle \text{Change variable} \rangle$

> $\equiv \langle \text{Substit. } x = y+2 \rangle\ (y+2)^3 - 6(y+2)^2 + 9(y+2) - 4 = 0$
>
> $\equiv \langle \text{Expand} \rangle\ (y^3 + 6y^2 + 12y + 8) - (6y^2 + 24y + 24) + (9y + 18) - 4 = 0$
>
> $\equiv \langle \text{Simplify} \rangle\ y^3 - 3y - 2 = 0$
>
> $\equiv \langle \text{Change variable} \rangle$
>
> > $\equiv \langle \text{Substit. } y = u + \frac{1}{u} \rangle\ \left(u + \frac{1}{u}\right)^3 - 3\left(u + \frac{1}{u}\right) - 2 = 0$
> >
> > $\equiv \langle \text{Expand} \rangle\ \left(u^3 + 3u + \frac{3}{u} + \frac{1}{u^3}\right) - 3\left(u + \frac{1}{u}\right) - 2 = 0$
> >
> > $\equiv \langle \text{Simplify} \rangle\ u^3 - 2 + \frac{1}{u^3} = 0$
> >
> > $\equiv \langle \text{Solve} \rangle$
> >
> > > $\equiv \langle \text{Multiplication prop.} \rangle\ u^3(u^3 - 2 + \frac{1}{u^3}) = u^3(0)$
> > >
> > > $\equiv \langle \text{Simplify} \rangle\ u^6 - 2u^3 + 1 = 0$
> > >
> > > $\equiv \langle \text{Factor} \rangle\ (u^3 - 1)^2 = 0$
> > >
> > > $\equiv \langle \text{Zero factor} \rangle\ u^3 = 1$
> > >
> > > $\equiv \langle \text{Cube roots} \rangle\ u = 1 \vee u = \frac{-1 + i\sqrt{3}}{2} \vee u = \frac{-1 - i\sqrt{3}}{2}$
> >
> > $\equiv \langle \text{Back substit. } y \rangle\ y = 1 + \frac{1}{1} \vee y = \frac{-1 + i\sqrt{3}}{2} + \frac{1}{\frac{-1 + i\sqrt{3}}{2}} \vee y = \frac{-1 - i\sqrt{3}}{2} + \frac{1}{\frac{-1 - i\sqrt{3}}{2}}$
> >
> > $\equiv \langle \text{Simplify} \rangle\ y = 2 \vee y = -1 \vee y = -1$

$\equiv \langle \text{Back substit. } x \rangle\ x = 2 + 2 \vee x = -1 + 2 \vee x = -1 + 2$

$\equiv \langle \text{Addition} \rangle\ x = 4 \vee x = 1 \vee x = 1$

$\equiv \langle \text{Elimin. } \vee \rangle\ x = 4 \vee x = 1$

This exercise is based on a form that Raymond Boute, professor emeritus at Ghent University, calls *calculational* proof style.[43] Something of an N-column extension of 2 Two-Column, this format separates each step of the proof from its justifications, which are nested like subroutines of an algorithm. The added depth of this form of proof accommodates the notable increase in proof complexity. This proof by double 22 Substitution (abbreviated here by "Substit.") is a variant of Cardano's method, which is discussed more fully in 25 Open Collaborative and 88 Dialogue. Some of the innermost justifications provided in the angle brackets are really just hints behind which hide further steps and hierarchies (e.g., ⟨Simplify⟩ is a stand in for some sequence of algebraic operations).

The symbol $\mathbb{R}$, which also appeared earlier in 15 Matrices, denotes the set of real numbers, and $\vee$ means "or."

Boute's motivation for advocating this style is mainly pedagogical, but he also finds the dominant style of writing at the professional level wanting. He likens the style to the pre-symbolic era of algebra (see 7 Found) and quotes computer scientist Leslie Lamport: "The structure of mathematical proofs has not changed in 300 years.... Proofs are still written like essays, in a stilted form of ordinary prose."[44] As moved as I am by this rallying cry, I have to wonder whether the real problem is prose itself or merely the stilted form it so often takes.

19
Jargon

Theorem. *The truth value of the material implication that the algebraic equality of the univariate monic cubic quadrinomial* $x^3 - 6x^2 + 11x - 6$ *with the univariate linear binomial* $2x - 2$ *over the field of real numbers implies the exclusive disjunction* $x = 1$ *or* $x = 4$ *is* True.

Proof. Suppose the existentialization of a field element x satisfying the antecedent algebraic equality. By the commutative and distributive properties of the field, the application of the polynomial subtraction operator with subtrahend $2x - 2$ to the antecedent equality results in the algebraic equality of the univariate monic cubic quadrinomial $x^3 - 6x^2 + 9x - 4$ and the additive identity element. The ensuing factorization $(x-1)^2(x-4)$ of the additive identity element necessitates, according to the nonexistence of zero divisors in a field, the consequent exclusive disjunction. QED

Jargon

This proof was written by puffing up 1 One-Line with mathematical terminology and incidental details. Thus, our cubic becomes *the algebraic equality of the univariate monic cubic quadrinomial* $x^3 - 6x^2 + 9x - 4$ *and the additive identity element*.

Like any other kind of writing, it is easier to write mathematics badly than it is to write it well. Nevertheless, an examination of style using a 13 Reductio Ad Absurdum approach has its merits.[45]

20

Definitional

Theorem (a statement derived from premises rather than assumed). *The truth value* (the attribute assigned as the semantic value of a proposition) *of the material implication* (the logical connective between two statements p and q that is equivalent to not both p and not q) *that the algebraic equality* (relationship between two algebraic expressions which asserts that they evaluate to the same value) *of the univariate* (single-variable) *monic* (having a leading coefficient of 1) *cubic* (third degree) *quadrinomial* (a four-term polynomial) $x^3-6x^2+11x-6$ *with the univariate* (single-variable) *linear* (first degree) *binomial* (two-term polynomial) $2x-2$ *over the field* (a nonzero commutative division ring) *of real numbers* (the numbers that can be expressed by a possibly infinite decimal representation) *implies the exclusive disjunction* (the logical operation that outputs *True* only when inputs differ) $x=1$ *or* $x=4$ *is* True (one of the two truth values in the Boolean domain, the other being *False*).

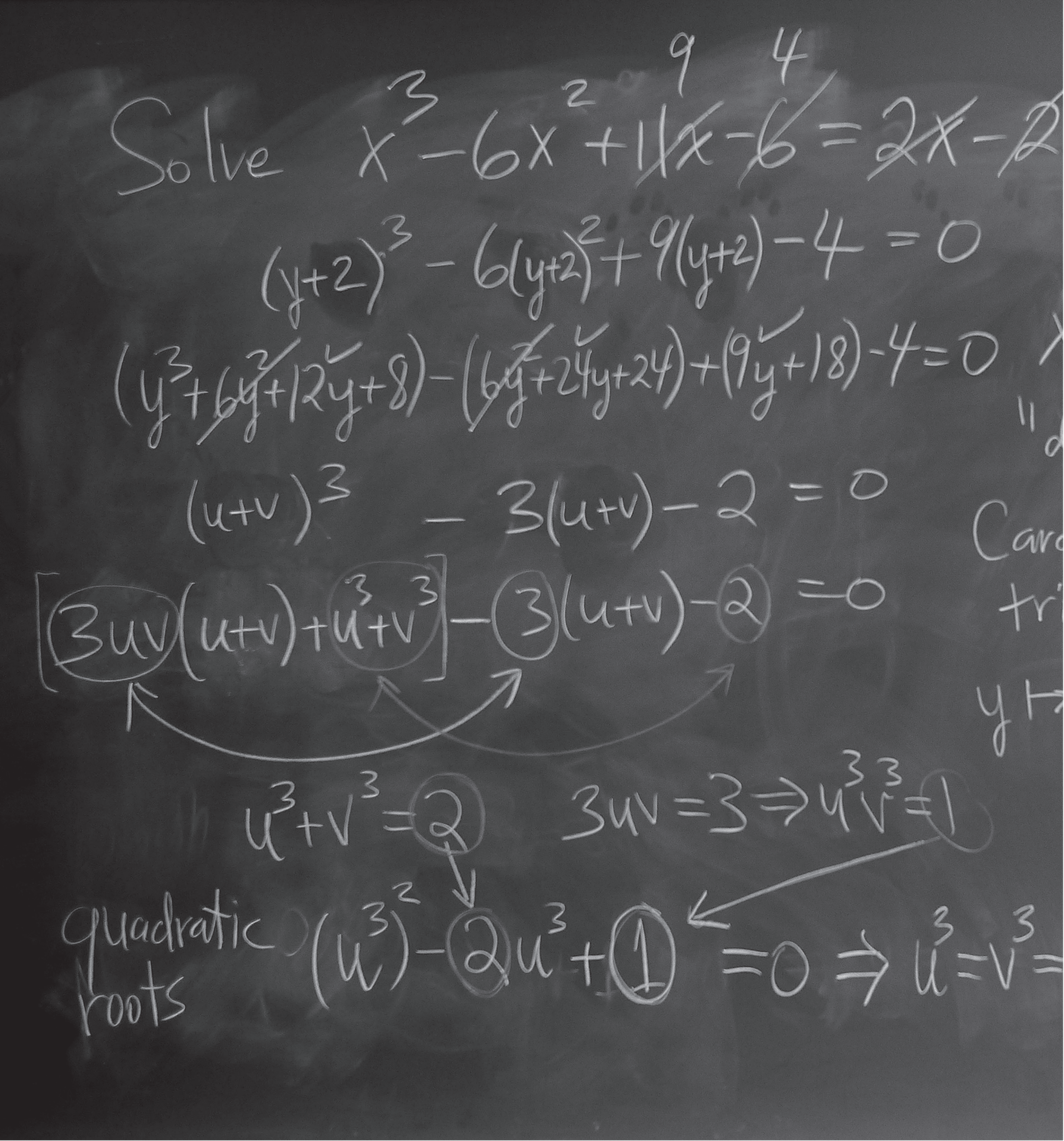
Proof (a chain of reasoning using rules of inference that leads to a desired conclusion). Suppose the existentialization *(a quantification which is interpreted as "there is at least one")* of a field element *(a member of a nonzero commutative division ring)* x satisfying the antecedent *(the first operand of a material implication)* algebraic equality *(relationship between two algebraic expressions which asserts that they evaluate to the same value).* By the commutative *(having the symmetry of a binary operation wherein the result does not depend on the order of the operands to which it is applied)* and distributive *(having the symmetry of an operation on a combination in which the result is the same as that obtained by performing the operation on the individual members of the combination, and then combining them)* properties of the field *(a nonzero commutative division ring)*, the application of the polynomial *(an expression that is a sum of terms, each term being a product of a constant and a non-negative power of a variable or variables)* subtraction operator *(the inverse of the addition operator)* with subtrahend *(the second operand of the subtraction operator which, when added to the difference, equals the minuend)* $2x-2$ to the antecedent *(the first operand of a material implication)* equality results in the algebraic equality *(relationship between two algebraic expressions which asserts that they evaluate to the same value)* of the univariate *(single-variable)* monic *(having a leading coefficient of 1)* cubic *(third degree)* quadrinomial *(a four-term polynomial)* x^3-6x^2+9x-4 and the additive identity element *(the element of a set that leaves other elements unchanged when added to them, often denoted as 0).* The ensuing factorization *(a decomposition into a product of factors, which when multiplied together give the original)* $(x-1)^2(x-4)$ of the additive identity element necessitates, according to the nonexistence of zero divisors *(nonzero elements in a ring that can be multiplied with another nonzero element in the ring to form a product equal to the*

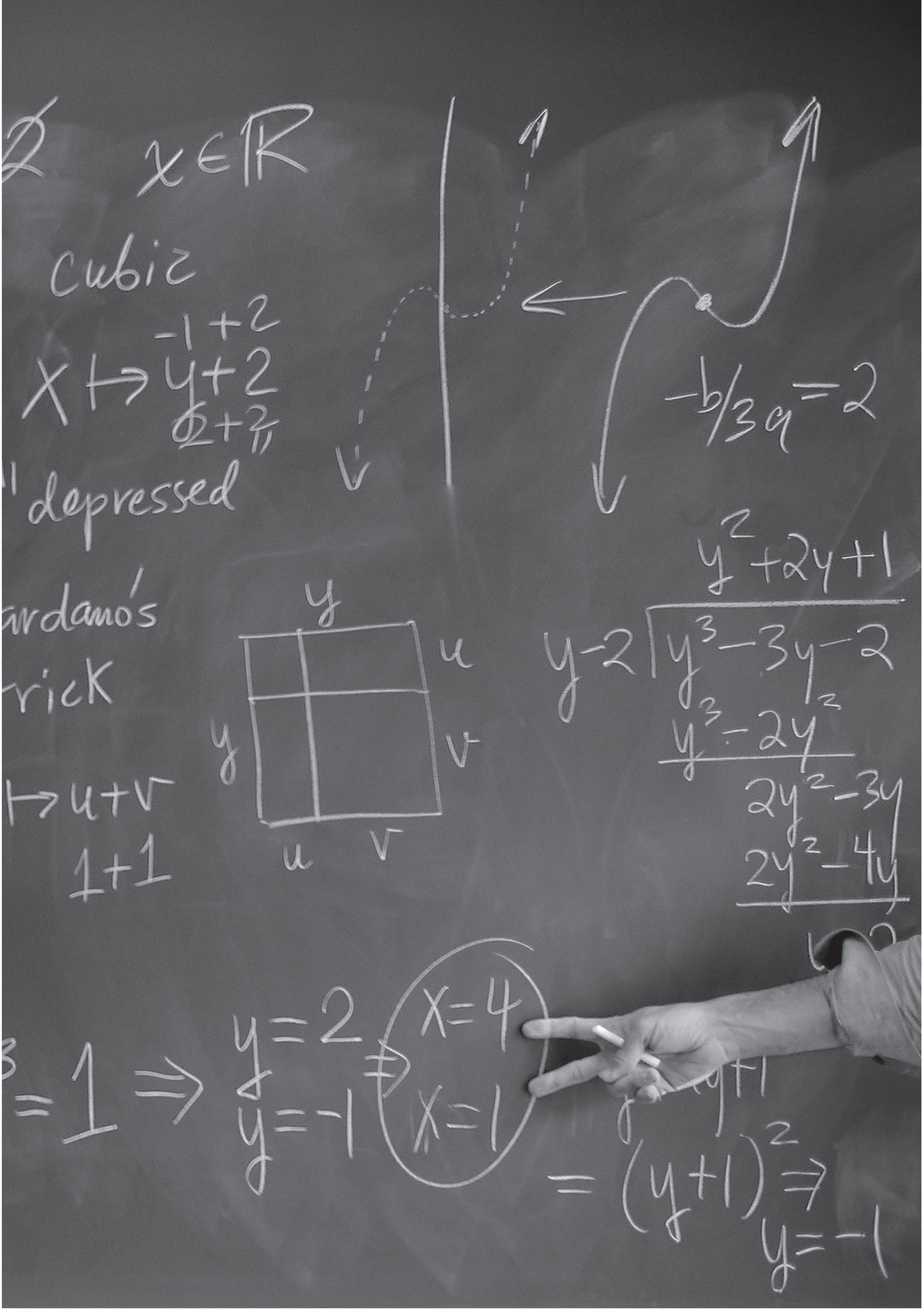
ring's zero element) in a field *(a nonzero commutative division ring)*, the consequent *(the second operand of a material implication)* exclusive disjunction *(the logical operation that outputs* True *only when inputs differ)*. QED *(quod erat demonstrandum / being what was required to prove)*

Unpacking the language of the statement of a theorem is usually an instructive exercise. When learning a new field of mathematics it's not unusual to have to look up the definition of a word that appears in the definition of a word that you've just looked up, and I contend that the inflationary process illustrated by this "larding" of 19 Jargon is not as much an exaggeration as it might appear.[45]

Sometimes this exercise by itself leads to a proof, which, we then say, "follows from definitions." One example is the theorem that every real number is the limit of a Cauchy sequence of rational numbers. What's a Cauchy sequence, you ask? Well, if we've *defined* a real number to be the limit of a Cauchy sequence (whatever that is) of rational numbers, then it doesn't matter. The proof follows from the definitions.

How is this even possible? Gian-Carlo Rota reminds us that "what an axiomatic presentation of a piece of mathematics *conceals* is at least as relevant to the understanding of mathematics as what an axiomatic presentation *pretends* to state."[46] In other words, the definitions of a mature theory are crafted with such ingenuity that alone they may be capable of the heavy lifting required to prove a theorem.


Solve x^3 − 6x^2 + 11x − 6 = 2x − 2
(y+2)^3 − 6(y+2)^2 + 9(y+2) − 4 = 0
(y^3 + 6y^2 + 12y + 8) − (6y^2 + 24y + 24) + (9y + 18) − 4 = 0
(u+v)^3 − 3(u+v) − 2 = 0
[3uv(u+v) + u^3 + v^3] − 3(u+v) − 2 = 0
u^3 + v^3 = 2
3uv = 3 ⇒ u^3v^3 = 1
quadratic roots
(u^3)^2 − 2u^3 + 1 = 0 ⇒ u^3 = v^3 =



21

Blackboard

Blackboard

More than a teaching aid, a blackboard is a medium for mathematics. Science historian Michael Barany and sociologist Donald MacKenzie argue that the material and performative qualities of chalk and board contribute to the discipline's characteristic rigor in that mathematical arguments on a "visually shared" blackboard "proceed step by step, and can be challenged by the audience at each step."[48] This degree of exposure, while illuminating, is no doubt a source of anxiety when called *to the board*. Some students would rather avoid doing math altogether.

When Yale University incorporated the nascent technology of blackboards into geometry exams in 1830 it sparked a literal rebellion that ended in the expulsion of over forty students.[49] Even if one knows how to hold a piece of chalk to prevent screeching (like a wand—not like a pen), effective board-work is a skill to be learned like any other in mathematics. I still remember being chastised by a prominent Japanese knot theorist for my improper use of the eraser when diagraming a knot at the board.

The proof displayed here uses Cardano's method, which is detailed in 25 Open Collaborative.

22

Substitution

Theorem. *If x is a real number and $x^3-6x^2+11x-6=2x-2$, then $x=1$ or $x=4$.*

Proof. By subtracting $2x-2$ from both sides of the given equation, we have the cubic equation $x^3-6x^2+9x-4=0$. This can be simplified by the substitution $x=y+1$:

$$\begin{aligned}0&=(y+1)^3-6(y+1)^2+9(y+1)-4\\&=(y^3+3y^2+3y+1)-(6y^2+12y+6)+(9y+9)-4\\&=y^3-3y^2\\&=y^2(y-3).\end{aligned}$$

Therefore y is 0 or 3. It follows that x is 1 or 4, as was to be shown. □

In literary terms, we might identify substitution with "metaphor," as the historian and philosopher of mathematics Reviel Netz does. "Mathematics," he says, "can only become truly interesting and original when it involves the operation of seeing something as something else."[50]

Substitution, in technical terms, refers to the transformation of an equation, integral, or other object by replacing every instance of a variable (e.g., x) by some other variable expression (e.g., $y+1$) with the intention of either simplifying the object or casting it into a more complex but tractable form. Having found a solution to a problem thus changed (e.g., $y=0$), the final step in the method of substitution is to translate the solution back in terms of the original variable (e.g., $x=1$). See 47 Clever for an example of a non-elementary substitution.

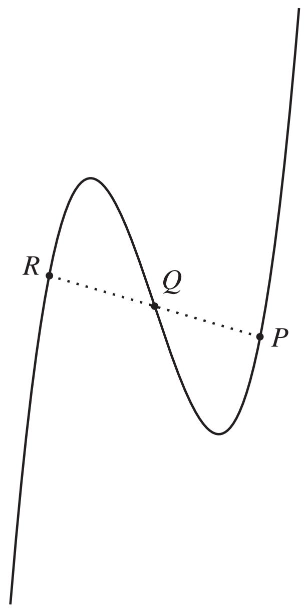
A sudden or unexpected change of terms can leave the reader as mystified as Goethe, who complained that mathematicians are "like a certain type of Frenchman: when you talk to them they translate it into their own language, and then it soon turns into something completely different."[51]

23

Symmetry

Theorem. *Let x be a real number. If $x^3 - 6x^2 + 11x - 6 = 2x - 2$, then x is 1 or 4.*

Proof. Subtracting $2x - 2$ from both sides of the equation shows that its solutions are the roots of the cubic curve $y = x^3 - 6x^2 + 9x - 4$. Every cubic curve is symmetric about its point of inflection Q; that is to say, P is a point on the curve if and only if $R = 2Q - P$ is as well. In particular, a root P of the cubic curve may be realized as the reflection $R = 2Q - P$.



In this case, $Q = (2, -2)$, which implies that the reflection of $P = (x, 0)$ is $R = 2(2, -2) - (x, 0) = (4 - x, -4)$. Setting $y = -4$, solve for the x-coordinate of R, x_R as follows:

$$\begin{aligned} x_R^3 - 6x_R^2 + 9x_R - 4 &= -4 \\ x_R^3 - 6x_R^2 + 9x_R &= 0 \\ x_R(x_R - 3)^2 &= 0. \end{aligned}$$

Hence x_R is 0 or 3. But $x_R = 4 - x$, and x is 1 or 4 as claimed. $\square$

Whenever possible, mathematicians look to see if a problem will divide along a line or axis of symmetry because it means there's a good chance that a solution to one part of the problem will extend, "by reflection" or "by rotation" as it were, to the problem as a whole. In this case, a symmetry about the inflection point—the point where the graph changes its sense of curving, from downward curving to upward curving—is exploited to reason about the desired solutions. Why is every cubic curve symmetric about its point of inflection?

24 Another Symmetry offers a more abstract idea of symmetry.

24

Another Symmetry

Theorem. *Let $x \in \mathbb{R}$. If $x^3 - 6x^2 + 11x - 6 = 2x - 2$, then $x = 1$ or $x = 4$.*

Proof. Suppose x_1, x_2, x_3 are the roots of the cubic polynomial

$$f(x) = x^3 - 6x^2 + 9x - 4.$$

Consider an associated polynomial $g(y)$ whose roots are given by a linear combination

$$y = c_1x_1 + c_2x_2 + c_3x_3$$

for a fixed set of coefficients c_1, c_2, c_3 and which does not depend on the ordering of the roots x_1, x_2, x_3. It must have a root for each reordering or permutation of the roots. There are $3! = 6$ permutations of three things, and the roots of $g(y)$ are:

$$\begin{aligned}
y_1 &= c_1x_1 + c_2x_2 + c_3x_3 \\
y_2 &= c_1x_1 + c_2x_3 + c_3x_2 \\
y_3 &= c_1x_2 + c_2x_1 + c_3x_3 \\
y_4 &= c_1x_2 + c_2x_3 + c_3x_1 \\
y_5 &= c_1x_3 + c_2x_1 + c_3x_2 \\
y_6 &= c_1x_3 + c_2x_2 + c_3x_1.
\end{aligned}$$

This implies that $g = g(y)$ is a degree six polynomial, which would appear to be even more difficult to solve. However, we can choose the coefficients c_1, c_2, c_3 so that the exponents of y in the nonzero terms of g have powers equal to multiples of three. This means that the degree of the polynomial $g = g(z)$, where $z = y^3$, is only two, and y would be the cube root of the solution of a quadratic. A necessary condition for this to occur is that if $g(y) = 0$, then $g(\omega y) = 0 = g(\omega^2 y)$, where ω is a primitive third root of unity, that is ω is a complex number such that $\omega^3 = 1$ and $\omega \neq 1$. If $\omega y_1 = y_2$, then $\omega c_1x_1 + \omega c_2x_2 + \omega c_3x_3 = c_1x_1 + c_2x_3 + c_3x_2$. But a comparison of the coefficients in x_1 reveals the contradiction $\omega = 1$. Hence $\omega y_1 \neq y_2$, and a similar arguments shows that $\omega y_1 \neq y_3$ and $\omega y_1 \neq y_6$. Suppose $\omega y_1 = y_4$, then

$$\omega c_1x_1 + \omega c_2x_2 + \omega c_3x_3 = c_1x_2 + c_2x_3 + c_3x_1.$$

By comparing coefficients we see $\omega c_1 = c_3$, $\omega c_2 = c_1$, $\omega c_3 = c_2$, or

$$c_2 = \omega^2 c_1 \qquad c_3 = \omega c_1.$$

Thus by setting $c_1 = 1$, we have $c_2 = \omega^2$, $c_3 = \omega$, and we can express all six roots of g in terms of y_1 and y_2. Indeed, $y_3 = \omega^2 y_2$, $y_4 = \omega y_1$, $y_5 = \omega^2 y_1$, and $y_6 = \omega y_2$.

The polynomial $g(y)$ then takes the form

$$\begin{aligned}g(y) &= (y-y_1)(y-\omega y_1)(y-\omega^2 y_1)(y-y_2)(y-\omega y_2)(y-\omega^2 y_2)\\ &= (y^3-y_1^3)(y^3-y_2^3)\\ &= y^6-(y_1^3+y_2^3)y^3+y_1^3y_2^3,\end{aligned}$$

where the second equality follows from the identity $1+\omega+\omega^2=0$.

Notice that the coefficients $y_1^3+y_2^3$ and $y_1^3y_2^3$ of $g(y)$ are symmetric with respect to all permutations of the roots x_1, x_2, x_3. The fundamental theorem of symmetric polynomials implies that these coefficients can be expressed in terms of the elementary symmetric polynomials

$$E_1=x_1+x_2+x_3, \qquad E_2=x_1x_2+x_2x_3+x_1x_3, \qquad E_3=x_1x_2x_3.$$

According to Vietà's formulas, these elementary symmetric polynomials can be evaluated in terms of the coefficients of $f(x)$: $E_1=6$, $E_2=9$, $E_3=4$. A somewhat tedious but straightforward computation then yields

$$y_1^3+y_2^3=2E_1^3-9E_1E_2+27E_3=54, \qquad y_1^3y_2^3=(E_1^2-3E_2)^3=729.$$

Hence

$$g(y)=y^6-54y^3+729=(y^3-27)^2,$$

and we obtain that the roots of $g(y)$ are $\{3,3\omega,3\omega^2\}$, and that each such root has multiplicity two. By symmetry, we may assume that $y_1=y_2=3$. The roots of $f(x)$ are now the solutions of the linear system

$$\begin{aligned}y_1 &= x_1+\omega^2x_2+\omega x_3=3\\ y_2 &= x_1+\omega x_2+\omega^2x_3=3\\ E_1 &= x_1+x_2+x_3=6.\end{aligned}$$

Solving this system, one finds that $x_1=4$, $x_2=x_3=1$, and the proof is complete. $\square$

This solution of the cubic also exploits symmetry, but in a more subtle way than the last variation. The strategy is due to the eighteenth century Italian-born mathematician whom most consider to be a Frenchman, Joseph-Louis Lagrange. Instead of immediately reducing the problem of solving the cubic, it first doubles the degree of the polynomial to a degree six polynomial—known as the Lagrange resolvent—and then reduces this to a quadratic equation. Although other proofs (e.g., 25 Open Collaborative) do the same, this proof accounts for the doubling of degree and subsequent division by three in terms of the symmetry of the roots of a polynomial.

A polynomial in two or more variables is called a *symmetric polynomial* if any permutation of its variables leaves the polynomial unchanged. For example, the elementary symmetric polynomials up to degree three are E_1, E_2, E_3. In general, there are elementary symmetric polynomials up to any degree. The fundamental theorem of symmetric polynomials states (roughly) that every symmetric polynomial (of whatever degree) can be expressed as a unique polynomial in the elementary symmetric functions.[52]

This modern conception of symmetry is expressed most often in the language of *groups*, and Lagrange's work is central to the prehistory of group theory.[53]

25

Open Collaborative

Problem: Prove that if $x \in \mathbb{R}$ and $x^3 - 6x^2 + 11x - 6 = 2x - 2$, then $x = 1$ or $x = 4$.

1. 
Just to get things going, it's perhaps worth noting that, in standard form, the polynomial equation is equivalent to

$$x^3 - 6x^2 + 9x - 4 = 0.$$

Comment by Alpha, June 25 @ 5:03pm | Reply

1.1. 
You could also express it in Horner's (another "standard") form

$$((x-6)x+9)x-4=0.$$

Comment by Beta, June 25 @ 5:29pm | Reply

2. 
I'm sure others have already tried this, but I'm just wondering whether it can't be factored

$$x(x^2 - 6x + 9) - 4 = 0$$
$$x(x-3)^2 = 4.$$

Comment by Gamma, June 25 @ 5:35pm | Reply

2.1. 
I was just thinking along the same lines, but thought it would be cheating to use the factor theorem, which, knowing the solutions, gives

$$(x-1)(x-4)(x-4) = 0.$$

Comment by Delta, June 25 @ 5:36pm | Reply

2.1.1. 
Sorry, but why are there three solutions? And isn't 1 the repeated root?
Comment by Epsilon, June 25 @ 5:45pm | Reply

2.1.2. 
That follows from the Fundamental Thm of Algebra, but using such a big hammer for this problem seems like another kind of cheating. Oops! You're right, should be $(x-1)(x-1)\ (x-4) = 0$.
Comment by Delta, June 25 @ 5:50pm | Reply

3. 
One usually solves quadratics that don't obviously factor by "completing the square." Is there such a thing as "completing the cube"?? Not sure this is a real strategy as yet.
Comment by Zeta, June 25 @ 5:59pm | Reply

3.1. The cubic binomial expansion is $(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$. Given that our quadratic coefficient is -6, we ought to take $a = -2$, so

$$x^3 - 6x^2 + 9x - 4 = (x-2)^3 - 3x + 4.$$

Seems like there's no guarantee that we can complete a cube, at least not by simply adding a constant.
Comment by Alpha, June 25 @ 6:19pm | Reply

4. Actually, I just noticed that the original equation factors very neatly

$$(x-1)(x-2)(x-3) = 2(x-1).$$

Comment by Gamma, June 25 @ 6:20pm | Reply

4.1. Canceling the common factor $(x-1)$ gives a quadratic with the desired solutions!
Comment by Epsilon, June 25 @ 6:22pm | Reply

5. I want to go back to Alpha's last comment. If you make the change of variable $y = x - 2$ then you do at least get a cubic without a square term,

$$y^3 - 3y - 2 = 0.$$

This seems important.
Comment by Zeta, June 25 @ 6:41pm | Reply

5.1. Tau calls this a "depressed cubic". Here's the relevant paper.
Comment by Eta, June 25 @ 6:44pm | Reply

5.1.1. Thanks Eta, I hadn't seen this before!
Comment by Zeta, June 25 @ 6:47pm | Reply

5.2. If you take $z = x + 1$, the linear and constant term drop out

$$z^3 - 3z = 0$$

and it's easily factored. Is this cheating?
Comment by Gamma, June 25 @ 6:51pm | Reply

5.3. I don't really have an idea here, but the 3 seems significant.

$$y^3 = 3y + 2$$

Can we use it to kill more terms?
Comment by Alpha, June 25 @ 6:52pm | Reply

5.3.1. Maybe this factored form of the cubic binomial expansion helps?

$$(x+a)^3 = x^3 + 3xa(x+a) + a^3$$

Comment by Beta, June 25 @ 7:22pm | Reply

5.3.2. Ah, yes! Let's take $y=u+v$, then

$$(u+v)^3 = 3uv(u+v) + u^3 + v^3$$

looks very much like our equation indeed

$$y^3 = 3y - 2.$$

Now we need to solve the system

$$\begin{aligned} uv &= 1 \\ u^3 + v^3 &= 2. \end{aligned}$$

Comment by Alpha, June 25 @ 7:25pm | Reply

5.3.3. That's a quadratic in u^3. Substitute $v=1/u$ into the second equation

$$\begin{aligned} u^3 + \frac{1}{u^3} &= 2 \\ u^6 - 2u^3 + 1 &= 0 \end{aligned}$$

with solution $u^3=1$, which implies $v=1$ and $y=2$. We have a solution!
Comment by Zeta, June 25 @ 7:28pm | Reply

5.3.4. Since $y=x-2$, that gives the root $x=4$.
Comment by Alpha, June 25 @ 7:29pm | Reply

5.3.5. Dividing out the original cubic by $(x-4)$ leaves $(x-1)^2=0$.
Comment by Zeta, June 25 @ 7:31pm | Reply

6. That's it, we have a proof!
Comment by Alpha, June 25 @ 7:32pm | Reply

Open Collaborative

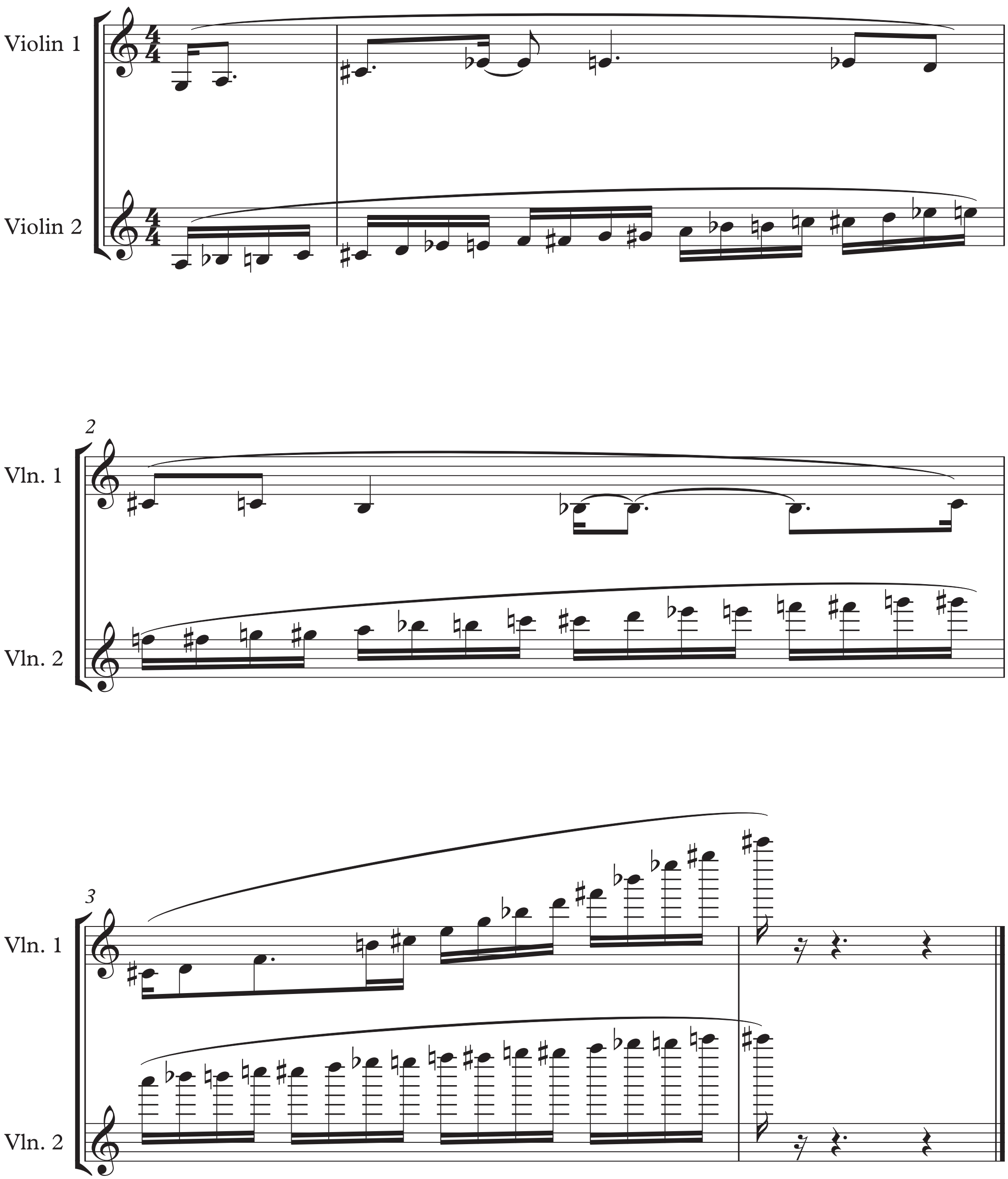
This style is based on the Polymath Project. In 2009, mathematician Timothy Gowers of Cambridge University used his blog to propose, and later conduct, an experiment to address the question, "Is massively collaborative mathematics possible?"[54] The first problem he posted was to find a new proof of a special case of the density Hales-Jewett theorem in combinatorics. Unexpectedly, this was achieved in just over five weeks, with twenty-seven people participating in the online discussion. "Anybody who had anything whatsoever to say about the problem could chip in," Gowers explained.[55] More recently, Polymath 8 (organized by another Fields Medalist, Terrence Tao) succeeded in refining Yitang Zhang's work on the twin primes conjecture.[56]

Besides its continuing practical value to mathematical research, the Polymath Project "shows vividly how ideas grow, change, improve and are discarded... and how even the best mathematicians can make basic mistakes and pursue many failed ideas."[57] Here Gowers is echoing Imre Lakatos, the twentieth century philosopher of mathematics and science whose rational reconstruction of the historical development of a mathematical concept, *Proofs and Refutations* reads very much like a proto Polymath project, albeit with a lot more kvetching.[58]

The small images appearing to the right of the comments are avatars that were created with permission by Scott Sherrill-Mix's WordPress plugins WP_Identicon and WP_MonsterID, which are based on software by Don Park and Andreas Gohr, respectively. The one exception is Tau's avatar, which is based on the 1546 frontispiece of Niccolò Tartaglia's *Quesiti et inventioni diverse*.

26

Auditory



The score represents each side of the equation by a violin; the first violin plays a melody that approximates the cubic function on the left side of the equation while the second violin plays a chromatic line that represents the linear function on the right side of the equation. The four and a quarter whole note duration of the piece corresponds to the interval $\frac{3}{4} \leq x \leq 4$. The range of the two functions over this interval has been divided into 54 semitones, which is the range of the continuously pitched classical musical instrument with the largest range—the violin—assuming a few extra high notes can be played off the fingerboard. The solutions to the equation occur when the two violins play the same note; that is, the C♯ on the first beat of the first full measure and the C♯ on the first beat of the last (partial) measure.

27

Algorithmic

Input
$x^3-6x^2+11x-6=2x-2$: cubic polynomial equation with two solutions;
x : symbol;

Output
the two solutions;

Needs
Left_side, *Right_side*, *Derivative*, *Remainder*, *Solve*, *Quotient*;

Local Variables
A, B, P, Q, R, x_1, x_2;

Begin
$A := Left\_side(x^3-6x^2+11x-6=2x-2)$;
$B := Right\_side(x^3-6x^2+11x-6=2x-2)$;
$P := A - B$;
$Q := Derivative(P, x)$;
while $Q \neq 0$ **do**
    $R := Remainder(P, Q, x)$;
    $P := Q$;
    $Q := R$;
$x_1 := Solve(P, x)$;
$x_2 := Solve(Quotient(A-B, (x-x_1)*(x-x_1)))$;
$Return([x_1, x_2])$;
End

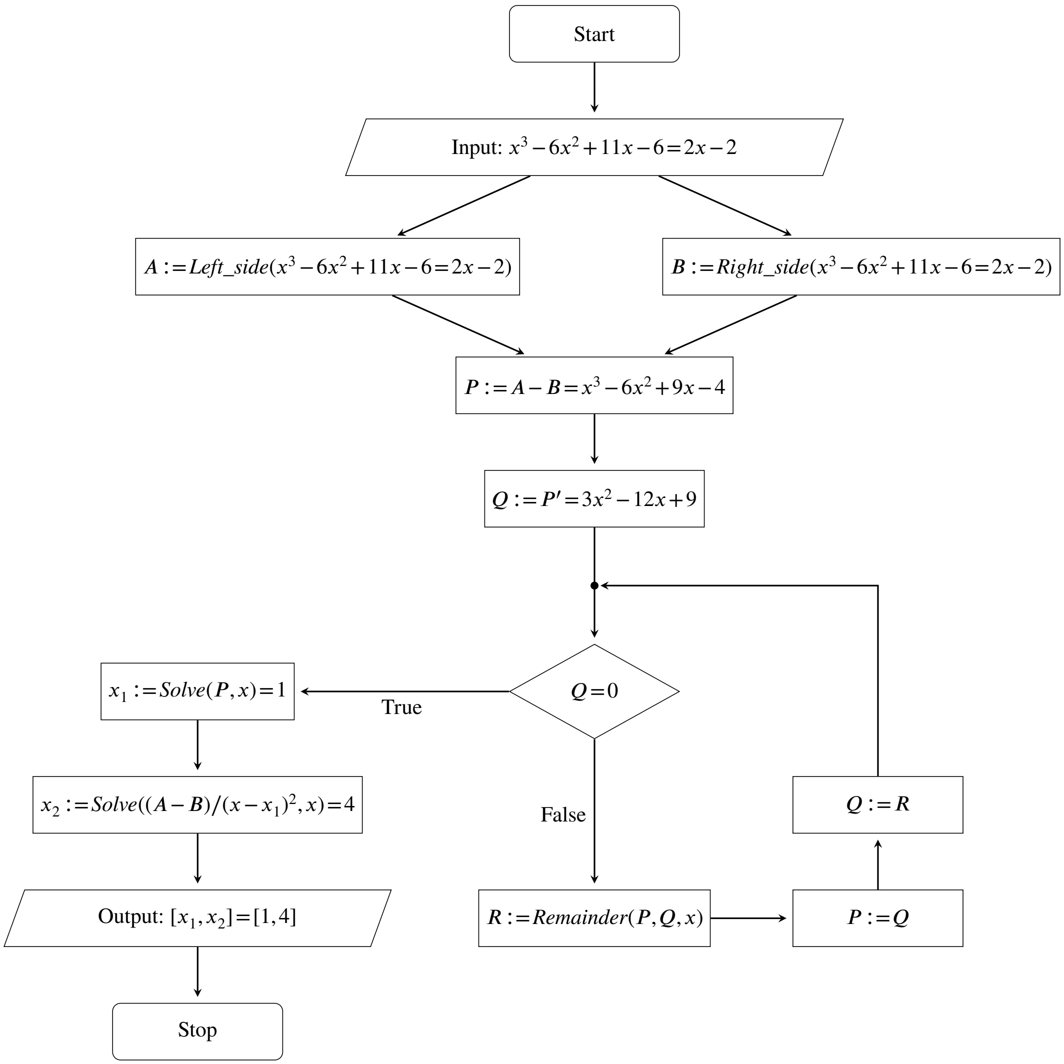
How would a computer solve a cubic? Interestingly, one of the first things that a computer algebra system will do when fed a polynomial equation is look for repeated roots, just like $x = 1$ in the case of our cubic. The key mathematical fact needed to do this comes from calculus: a repeated root of a polynomial P corresponds to a root shared with its derivative Q. If you noticed that the two figures in the 3 Illustrated proof cross at $x = 4$ but appear to be *tangent* at the repeated root $x = 1$, this is the reason why.

Finding a common root between two polynomials is a straightforward process, known as the Euclidean algorithm, and it is no more complicated than finding the largest whole number that evenly divides two given integers. This is accomplished by the "while" loop here. Once this subroutine finds the repeated root $x = 1$, it divides the cubic by $(x-1)^2$ to find the remaining solution.

In the same way that pseudo-code is a style of coding without using a specific programming language, mathematical pseudo-language is a style of computing without specifying a computer algebra system. This algorithm and the pseudo-language that I've used to convey it are based on *Computer Algebra and Symbolic Computation: Mathematical Methods* by Joel S. Cohen.[59]

28

Flow Chart



Flow Chart

A flow chart is “a device for visualizing a process as a means of improving it.”[60] In this case the process is the previous 27 Algorithm variation. The 59 Patented variation appropriates this flow chart. School teachers sometimes use flow charts as visual aids to students learning the process of solving an elementary algebraic equation and the basic steps of a proof by contradiction.[61] Other types of flow charts, including state diagrams and decision trees, commonly appear in applied mathematics contexts.

29

Model

As discussed in the 25 Open Collaborative proof and elsewhere, a change of variable allows us to reduce our cubic equation to the quadratic-free cubic

$$z^3 - 3z - 2 = 0.$$

In fact, every cubic equation admits the form

$$z^3 + pz + q = 0$$

for some choice of coefficients p and q. If we interpret the triple of numbers (p, q, z) as geometric coordinates (x, y, z), then it becomes possible to construct the *space* of solutions to all reduced cubics as the graph of the equation

$$z^3 + xz + y = 0.$$

The photograph shows a paper model of the surface defined by this equation with a hand pointing in the direction of the coordinates $(-3, -2, 2)$ and $(-3, -2, -1)$ that represent the two solutions 2 and -1 of our reduced cubic, for which $p = -3$ and $q = -2$.

Physical models of algebraic surfaces, most often made from plaster but also paper, wood, or string, enjoyed a broad distribution in the late nineteenth and early twentieth centuries. Mathematician Arnold Emch made a model of this surface sometime around 1935, though I wasn't able to confirm whether or not it is still in the collection of the University of Illinois where he was professor.[62]

This model was made by Sarah Dennis, an undergraduate student working on this project, by printing computer plots of cross sections of the region beneath the surface, cutting vertical notches in each, and interlocking the cross sections. Such "sliceform" models, as they've come to be called,[63] were likely a preliminary step in the fabrication of plaster models.[64]

30

Formulaic

To find the solution of $x^3-6x^2+11x-6=2x-2$, first rewrite the equation in standard form as $x^3-6x^2+9x-4=0$. The root of a cubic polynomial $a\,x^3+b\,x^2+c\,x+d$ is given by Cardano's formula:

$$x=\sqrt[3]{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)+\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}}$$
$$+\sqrt[3]{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)-\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}}-\frac{b}{3a}.$$

Given the coefficients, $a=1$, $b=-6$, $c=9$, $d=-4$, the formula provides the following solution:

$$x=\sqrt[3]{\left(\frac{-(-6)^3}{27(1)^3}+\frac{(-6)(9)}{6(1)^2}-\frac{-4}{2(1)}\right)+\sqrt{\left(\frac{-(-6)^3}{27(1)^3}+\frac{(-6)(9)}{6(1)^2}-\frac{-4}{2(1)}\right)^2+\left(\frac{9}{3(1)}-\frac{(-6)^2}{9(1)^2}\right)^3}}$$
$$+\sqrt[3]{\left(\frac{-(-6)^3}{27(1)^3}+\frac{(-6)(9)}{6(1)^2}-\frac{-4}{2(1)}\right)-\sqrt{\left(\frac{-(-6)^3}{27(1)^3}+\frac{(-6)(9)}{6(1)^2}-\frac{-4}{2(1)}\right)^2+\left(\frac{9}{3(1)}-\frac{(-6)^2}{9(1)^2}\right)^3}}-\frac{-6}{3(1)}$$
$$=\sqrt[3]{1}+\sqrt[3]{1}+2$$
$$=4.$$

The difficulty remembering Cardano's formula isn't just that it's so long, there's also a subtlety at work in the cube roots. Just as one needs to account for both positive and negative square roots in the quadratic cousin to this formula, there are additional solutions resulting from different *complex* cube roots. This subtlety, which was ignored by the author of this exercise, gets a fuller treatment in 31 Counterexample. Here is a complete formula for each root x_n of the general cubic equation, where $n = 0, 1, 2$:

$$x_n = \left(\frac{-1+\sqrt{-3}}{2}\right)^n \sqrt[3]{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)+\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}}$$
$$+\left(\frac{-1+\sqrt{-3}}{2}\right)^{2n} \sqrt[3]{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)-\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}}$$
$$-\frac{b}{3a}.$$

The formula is named for the author of the 7 Found proof.

31

Counterexample

Claim. *If* $x^3-6x^2+11x-6=2x-2$, *then Cardano's formula implies that* $x=4$.

Counterexample. Cardano's formula returns the following equation

$$x=\sqrt[3]{1}+\sqrt[3]{1}+2,$$

and by taking the value of $\sqrt[3]{1}$ to be 1 we obtain the given solution $x=4$. However, 1 is not the only number whose cube equals 1. Observe

$$\begin{aligned}\left(\frac{-1+\sqrt{-3}}{2}\right)^3&=\frac{(-1+\sqrt{-3})(-1+\sqrt{-3})^2}{2^3}\\&=\frac{(-1+\sqrt{-3})(-2-2\sqrt{-3})}{8}\\&=\frac{2-2(-3)}{8}\\&=1.\end{aligned}$$

A similar exercise also shows that

$$\left(\frac{-1-\sqrt{-3}}{2}\right)^3=1.$$

Combining these two complex cube roots of unity yields a second solution

$$x=\sqrt[3]{1}+\sqrt[3]{1}+2=\frac{-1+\sqrt{-3}}{2}+\frac{-1-\sqrt{-3}}{2}+2=1,$$

and the claim is refuted.

Given that algebraists in the Renaissance were not yet granting negative quantities full status as numbers, it may seem unreasonable to expect Cardano to have taken up *square roots* of negative numbers. And yet he does. In chapter 37 of the *Ars Magna* he first notes that the "true" solution $x = 5$ of the equation $x^2 = x + 20$ becomes a negative solution when the equation is "turned around" as $x^2 + x = 20$. Further into the same chapter he presents the following oft-cited discussion (translation by David Eugene Smith):

> If it should be said, Divide 10 into two parts the product of which is 30 or 40, it is clear that this case is impossible. Nevertheless, we will work thus: We divide 10 into two equal parts, making each 5. These we square, making 25. Subtract 40, if you will, from the 25 thus produced, as I showed you in the chapter on operations in the sixth book, leaving a remainder of -15, the square root of which added to or subtracted from 5 gives parts the product of which is 40. These will be $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$.... Putting aside the mental tortures involved, multiply $5 + \sqrt{-15}$ by $5 - \sqrt{-15}$, making $25 - (-15)$ which is $+15$. Hence this product is 40.[65]

32
Another Counterexample

Claim. *If* $x^3-6x^2+11x-6=2x-2$*, then* $x=1$ *or* $x=4$.

Counterexample. Subtracting $2x-2$ from both sides of the equation puts the cubic polynomial in standard form $x^3-6x^2+9x-4=0$, which factors as

$$(x-1)^2(x-4)=0.$$

This implies that $x-1=0$ or $x-4=0$, and thus $x=1$ or $x=4$ as claimed, but only under the assumption that x belongs to a set of numbers X that satisfies the zero product property:

$$\text{for all } a,b\in X,\ \text{if } ab=0 \text{ then } a=0 \text{ or } b=0.$$

This need not be the case, as for example when X is the (finite) set of remainders after dividing an integer by 12

$$X=\{0,1,2,3,4,5,6,7,8,9,10,11\}.$$

Define the sum, difference, and product of any pair of remainders to be the remainder of their usual sum, difference, and product after dividing by 12. Now we see that when $x=7$ the factorization above becomes:

$$(x-1)^2(x-4)=(7-1)^2(7-4)=108.$$

Since 12 divides 108 evenly this product corresponds to the remainder 0 in X. Altogether we have found three solutions to the equation, $x_1=1$, $x_2=4$, and $x_3=7$, and the claim is refuted.

Another Counterexample

Arithmetic on the set of remainders after dividing by twelve (or *modulo* 12) can be thought of as "clock arithmetic," as it functions in a similar way to the hours on a clock face.

The claim in this exercise could be viewed as an improvement made in response to the first criticism, 31 Counterexample. How could this claim be further refined to take into account this second counterexample? Typically one would limit the possible values of the variable x to the set of integers or rational numbers or real numbers, as many of the theorems in this collection do. (The least restrictive set of numbers that satisfies the zero-product property is called a *domain*.)

As mentioned in the discussion following 25 Open Collaborative, Imre Lakatos was one of the first philosophers of mathematics to systematically study the back and forth between claims and counterexamples that is typical of the historical development of mathematics.[66] Another feature of Lakatos's analysis of counterexamples that I have imitated here is the way they can emerge from criticism of *different* proofs of the *same* claims. (Note that the counterexample of this claim also refutes the claim in 31 Counterexample.) As a result, Lakatos points out, it is often possible to recognize in the various assumptions listed at the start of a theorem—or an as-yet-unfalsified claim—the traces of the multiple *failed* proof ideas from which it was wrought.

33

Calculus

Theorem. *Define $f : \mathbb{R} \to \mathbb{R}$ to be the function $f(x) = x^3 - 6x^2 + 9x - 4$. If $f(x) = 0$, then $x = 1$ or $x = 4$.*

Proof. Let's construct the Taylor series of f about the first of the two purported roots, $x = 1$. Compute the derivatives of f: $f'(x) = 3x^2 - 12x + 9$, $f''(x) = 6x - 12$, $f'''(x) = 6$, and $f^{(n)} = 0$ for $n \geq 4$. Hence

$$\begin{aligned} f(x) &= f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots \\ &= 0 + 0 - 3(x-1)^2 + (x-1)^3 + 0 \\ &= (x-1)^2(-3 + x - 1) \\ &= (x-1)^2(x-4). \end{aligned}$$

Thus, the roots of f are 1 and 4, as claimed.

American mathematician William Thurston's 1994 article "On Proof and Progress in Mathematics" outlines a list of different conceptualizations of the derivative.[67] It reads like an assignment for another set of exercises in style:

1. Infinitesimal
2. Symbolic
3. Logical
4. Geometric
5. Rate
6. Approximation
7. Microscopic

$\vdots$

37. Langrangian section of a cotangent bundle.

"Mathematics in some sense has a common language: a language of symbols, technical definitions, computations, and logic," Thurston observed. "This language efficiently conveys some, but not all, modes of mathematical thinking," he warned; "Unless great efforts are made to maintain the tone and flavor of the original human insights, the differences start to evaporate as soon as the mental concepts are translated into precise, formal and explicit definitions."[68]

The first draft of this proof applied the product rule exemplifying the symbolic way of thinking. The derivative here is conceived as an approximation to the cubic. Thanks to Amy Feaver of The King's University in Edmonton for suggesting this revision.

34

Medieval

Suppose that the intensity of a quality is as the cube of its extension and 9 times that less 6 times its square. It will be demonstrated that when this quality achieves an intensity of 4, its extension is 1 or 4. One speaks of such a quality as difformly difform, and it is necessary to have recourse to a speculative mensuration of the curved figures. Our speculation will proceed in accordance with the method of double false position. To begin, put arbitrarily that the extension is 1. Therefore the cube and 9 times the extension less 6 times its square, namely 1 and 9 less 6, is exactly equal to 4. Indeed 1 is a true solution, as claimed. Next put arbitrarily that the extension is 2. The cube and 9 times it less 6 times its square, namely 8 and 18 minus 24, results in 2. This differs from 4, the true value, by minus 2, therefore this position is false. Thus for the extension put 12. The sum of the cube and 9 times it less 6 times the square, namely 1728 and 108 minus 864, results in 972. This second position is also false. The difference in the approximations is their sum, 2 and 972, namely 974, since one error is minus and the other plus. The question that now arises as to what to add to the first position so as to decrease the difference between the value of minus 2 that resulted and 4, the true number. Suppose that the quality were uniformly difform over this difference. Then multiply this difference by the difference between the two positions and divide by the difference in the approximations that they produced, namely the 2 by the 10, and divide by the 974. The quotient is $\frac{2}{97}$, therefore augment the extension by an amount of one unit to 3. This position is similarly false. The sum of the cube and 9 times it less 6 times the square, namely twice 27 minus 54, yields no intensity at all. The second position, namely 12, resulted in an intensity much greater than the true one; therefore halve the extension, and put 6 for the new second position. The sum of the cube and 9 times it less 6 times the square, namely 216 and 54 minus 216, now results in 54. This new second position is also false. Proceeding as before, multiply the difference by the difference between the two positions; divide by the difference in the approximations that they produced, namely the 4 by the 3; and divide by the 54. The quotient is $\frac{2}{9}$, whence the extension is augmented one unit again, this time from 3 to 4. Now the cube and 9 times the extension less 6 times its square, namely 64 and 36 less 96, is exactly equal in intensity to 4. Hence 4 is the second true solution.

This proof is a pastiche of two medieval sources: Chapter 13 "on the method elchataym and how with it nearly all problems of mathematics are solved" of the *Liber abaci* (*Book of Calculation*, 1202) by Leonardo of Pisa, a.k.a. Fibonacci,[69] and the *Tractatus de configurationibus qualitatum et motuum* (*A Treatise on the Configuration of Qualities and Motions*, ca. 1370) by the scholastic philosopher Nicole Oresme.[70]

The method of elchataym or double false position is known today as linear interpolation. Given the task of estimating the solution x to the equation $f(x)=y$ for some continuous function f, we make two guesses x_1, x_2, calculate their values $y_1=f(x_1)$, $y_2=f(x_2)$, and solve the equation of the line connecting the "false positions":

$$x=x_1+(y_1-y)\frac{x_2-x_1}{y_2-y_1}.$$

Remarkably, Fibonacci obtained the real root of the cubic $x^3+2x^2+10x=20$ to an accuracy of nine decimal digits, possibly by repeated application of this method.[71] That he refers to the method with the Arabic "elchataym" acknowledges the medieval Islamic mathematics from which he probably learned it. An earlier instance of the method appears in the ancient Chinese text, *The Nine Chapters on the Mathematical Art*.[72]

Oresme's analysis of the intensity of a quality (or velocity) in terms of extension is a precursor of the modern notion of a function. In this correspondence a linear function with nonzero slope is called a *uniform difform*; a nonlinear function, a *difform deformity*.

To read the text of this variation in a less mesmerizing font face, see 35 Typeset.

35

Typeset

```
\documentclass[11pt]{book}
\usepackage{multicol,yfonts,lettrine}%needs yinitas.mf
\begin{document}
\begin{center}\begin{multicols}{2}
\textfrak{\begin{spacing}{1.25}\large
\lettrine[lines=3]{S}{uppose} that the intensity of a quality is
as the cube of its extension and 9 times that less 6 times its
square. It will be demonstrated that when this quality achieves
an intensity of 4, its extension is 1 or 4. One speaks of such a
quality as difformly difform, and it is necessary to have recourse
to a speculative mensuration of the curved figures. Our
speculation will proceed in accordance with the method of double
false position.
%
To begin, put arbitrarily that the extension is 1. Therefore the
cube and 9 times the extension less 6 times its square, namely 1
and 9 less 6, is exactly equal to 4. Indeed 1 is a true solution,
as claimed.
%
Next put arbitrarily that the extension is 2. The cube and 9 times
it less 6 times its square, namely 8 and 18 minus 24, results in
2. This differs from 4, the true value, by minus 2, therefore this
position is false. Thus for the extension put 12. The sum of the
cube and 9 times it less 6 times the square, namely 1728 and 108
minus 864, results in 972. This second position is also false. The
difference in the approximations is their sum, 2 and 972, namely
974, since one error is minus and the other plus.
%
The question now arises as to what to add to the first position
so as to decrease the difference between the value of minus 2 that
resulted and 4, the true number. Suppose that the quality were
uniformly difform over this difference. Then multiply this
difference by the difference between the two positions and divide
by the difference in the approximations that they produced, namely
the 2 by the 10, and divide by the 974. The quotient is $\frac{\
textfrak{2}}{\textfrak{97}}$, therefore augment the extension by
an amount of one unit to 3. This position is similarly false. The
sum of the cube and 9 times it less 6 times the square, namely
twice 27 minus 54, yields no intensity at all. The second
```

```
position, namely 12, resulted in an intensity much greater than
the true one; therefore halve the extension, and put 6 for the new
second position. The sum of the cube and 9 times it less 6 times
the square, namely 216 and 54 minus 216, now results in 54. This
new second position is also false.
%
Proceeding as before, multiply the difference by the difference
between the two positions; divide by the difference in the
approximations that they produced, namely the 4 by the 3;
and divide by the 54. The quotient is $\frac{\textfrak{2}}{
\textfrak{9}}$, whence the extension is augmented one unit again,
this time from 3 to 4. Now the cube and 9 times the extension less
6 times its square, namely 64 and 36 less 96, is exactly equal in
intensity to 4. Hence 4 is the second true solution.
\end{spacing}}
\end{multicols}
\end{document}
```

This exercise provides the electronic typesetting source code for 34 Medieval.

When the idea of adapting *Exercises in Style* to a mathematical proof entered my head, I tried to block the infection by convincing myself that someone else had already done it. Unfortunately, the only evidence I could find at the time was an outline of six potential directions for exercises in style by a one-time director at Wikipedia.fr who goes by the name Ellisllk.[73] Eight years later, however, I learn that in fact someone else has by then done it. *Rationnel mon Q: 65 exercices de styles* by Ludmila Duchêne and Agnès Leblanc is a hilarious and skillful rendering of the proof of the irrationality of $\sqrt{2}$ into all manner of styles.[74] Interestingly, there are only a few of their styles that overlap with ones I had written. One example of their plagiarism by anticipation, as the Oulipo would say, is "Irrationalité de $\sqrt{2}$ (source)." Like the example here, it reveals the TEX source code that produces the typeset documents. Nearly everyone in the (international!) math and science community has come to rely on TEX (often pronounced "tek") since computer scientist and mathematician Donald Knuth released the open source typesetting system in 1978.

The high quality output and relatively small file size of `.tex` source files has brought about a fundamental change in mathematical publishing and research (see 37 Preprint). As an example of the source code for modern mathematical notation, here is the code for Cardano's formula in 30 Formulaic:

```
{\small
\begin{align*}
\begin{split}
x=&\sqrt[\leftroot{-1}\uproot{2}\scriptstyle 3]
{\left( \frac{-b^3}{27a^3} + \frac{bc}{6a^2} -
\frac{d}{2a}\right) + \sqrt{\left( \frac{-b^3}{27a^3}
+ \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 +
\left(\frac{c}{3a} - \frac{b^2}{9a^2} \right)^3}}
\\&+
\sqrt[\leftroot{-1}\uproot{2}\scriptstyle 3]
{\left( \frac{-b^3}{27a^3} + \frac{bc}{6a^2}
- \frac{d}{2a}\right) - \sqrt{\left( \frac{-b^3}{27a^3}
+ \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 +
\left(\frac{c}{3a} - \frac{b^2}{9a^2} \right)^3}}
- \frac{b}{3a}.
\end{split}
\end{align*}
}
```

36

Social Media

Girolamo Cardano
@realCardano

Cube & 9 times first power equal 6 times square & 4 solved by reduction to @delferro's equation arxiv.org/abs/4307.1160 #cubic #tartaglia

11:40 AM - 28 Jul 1543

14 37 99

Despite its reputation as a profession of lone geniuses or as a school subject for the socially awkward, mathematics does have a presence on social media, including Twitter. (Of course, as a technology, social media is in part a *product* of mathematics.) According to a 2014 article in *Science*, there were two mathematicians among the top 50 “science stars of Twitter.”[75] Marcus du Sautoy (@MarcusduSautoy) of the University of Oxford ranked #19 with 34,200 followers and 3,555 tweets, while John Allen Paulos (@JohnAllenPaulos) of Temple University came in at #43 with 14,000 followers and 4,144 tweets. Both were dwarfed by comparison with the astrophysicist and #1 science tweeter Neil deGrasse Tyson (@neiltyson) of the Hayden Planetarium, who had 2.4 million followers at the time. The 137-character message here was partially modeled on tweets by Mathematics Papers (@MathPaper), which posts links to new mathematics submissions to the preprint (arXiv) server. For the linked abstract, see 37 Preprint.

John McCleary’s *Exercises in (Mathematical) Style: Stories of Binomial Coefficients* appeared as I was finalizing this manuscript. It is driven more by mathematical content, if I understand its intentions correctly, but has some direct overlaps with these proofs as well. In particular, his chapter “Tweets” demonstrates a group of tweeters working through a proof.

Perhaps the time is ripe to reconstitute the Workshop for Potential Mathematics or Oumathpo. According to the *Oulipo Compendium*, “the principle objective of the [Oumathpo] was to make available to mathematics—in return for the structures that mathematics had contributed to literature (especially to the Oulipo)—mathematical applications of what had heretofore been purely literary procedures.”[76] Its members included Queneau, Le Lionnais, Roubaud, Berge, Paul Braffort, Georg Kreisel, Pierre Samuel, Gian-Carlo Rota, and Stanislaw Ulam.[77]

37

Preprint

```
----------------------------------------------------------------\\
arXiv: 4307.1160
Date: Wed, 28 Jul 1543 09:04:16 GMT   (11kb)

Title: On a cube and first power equal to a square and number
Authors: Girolamo Cardano
Categories: math.AG
Comments: 4 pages, 1 figure
\\
There has been important progress toward solving cubic equations
since the pioneering work of Khayyam. In this century, a general
method for obtaining solutions in the case of a cube and first
power equal to a number was provided by del Ferro. In the present
paper we compute all solutions of the cubic $x^3+9x=6x^2+4$ by
transforming it into a quadratic-free cubic. There follows a
discussion of certain derivative quartic equations. Additional
applications include compound interest, profit on repeated business
trips, and the proper distribution of monies among soldiers.
\\(http://arxiv.org/abs/4307.1160, 11kb)
```

Preprint

It's not uncommon for different mathematicians, working independently, to discover the same object or prove the same theorem. If authors want to assert priority of—or simply share—their work before it makes its way through the peer review process and appears in print (anywhere from six months to a year or longer), they will post an electronic preprint of it on the preprint archive at arxiv.org. (For a pre-internet method of ensuring priority of scientific discovery, see 80 Paranoid.)

This exercise takes the form of the daily e-mail alerts from the arXiv (pronounced "archive") server. Subscribers can choose from any of thirty-two different subject categories, including, for example, Algebraic Geometry (math.AG). In the year 2016, there were 32,553 preprints posted to the Mathematics section of the arXiv.[78]

38

Parataxis

$x^3 - 6x^2 + 11x - 6 = 2x - 2.$

$x^3 - 6x^2 + 9x - 4 = 0.$

$(y+2)^3 - 6(y+2)^2 + 9(y+2) - 4 = 0.$

$y^3 - 3y - 2 = 0.$

$(2z)^3 - 3(2z) - 2 = 0.$

$4z^3 - 3z = 1.$

$4(\cos\theta)^3 - 3(\cos\theta) = 1.$

$4\cos^3\theta - 3\cos\theta = \cos 3\theta.$

$\cos 3\theta = 1.$

$\theta = 0, \frac{2\pi}{3}.$

$z = -\frac{1}{2}, 1.$

$y = -1, 2.$

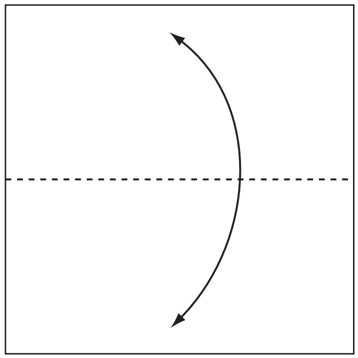
$x = 1, 4.$

See 47 Clever and its note for some details about the steps of this proof.

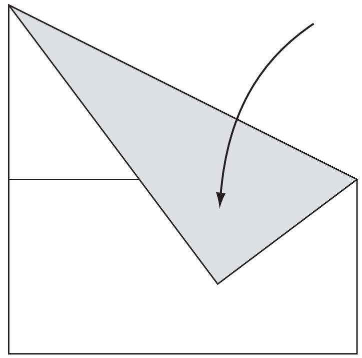
> Étienne Ghys, a mathematician at École Normale Supérieure de Lyon in France, recalled spending six months trying to understand the results of [Marina Ratner's] dynamics research to present them at a seminar. When he discussed the papers with her, he told her that he had the feeling that she had written the papers not for other mathematicians to understand but mainly to convince herself that the theorems were correct. Dr. Ghys said Dr. Ratner replied: "Yes! Exactly! You understood why and how I write mathematics."
>
> *The New York Times*[79]

39

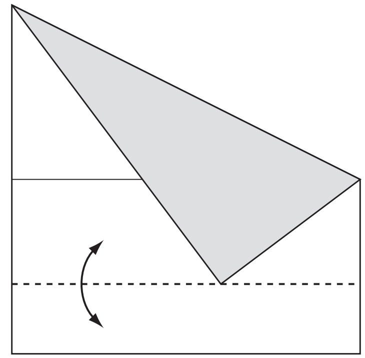
Origami



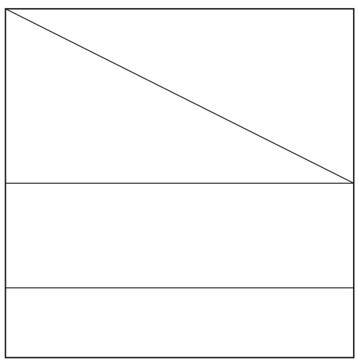
1. Fold in half to make crease and fold back



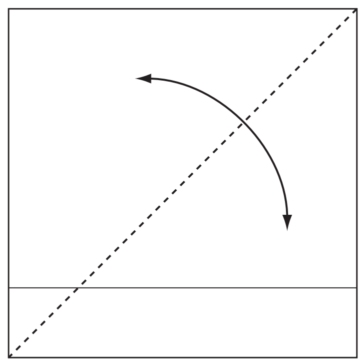
2. Make a diagonal fold from corner to center crease



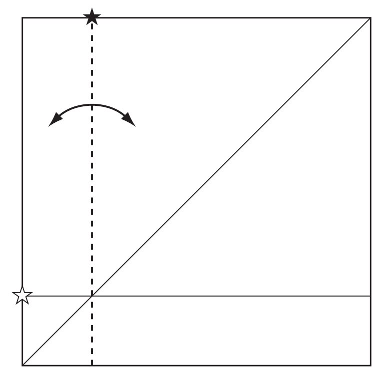
3. Fold horizontally to make crease and fold back



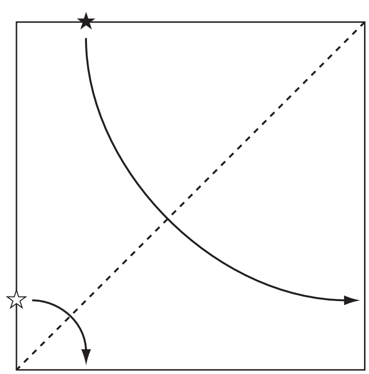
4. Unfold

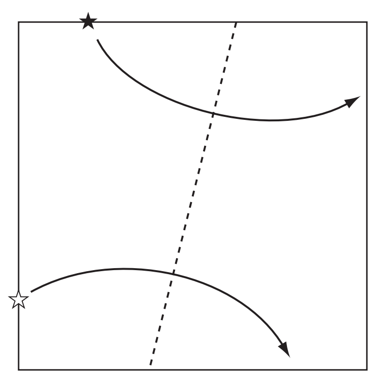


5. Fold along diagonal to make crease and fold back

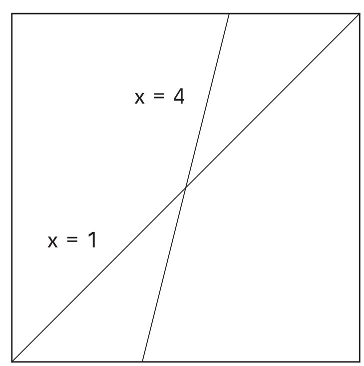


6. Fold vertically to make crease and fold back





7. Fold ☆ to meet bottom side and ★ to meet right side
The slope of each crease is a root of $x^3 - 6x^2 + 9x - 4$



8. Finished

Any length that is constructible by straightedge and compass (see 12 Ruler and Compass) can be constructed by folding paper, without need of any tools beyond hand and eye.[80] Furthermore, Italian mathematician Margharita Piazolla Beloch discovered that paper folding is, strictly speaking, a stronger tool because it can solve any cubic equation.[81] Beloch showed this by demonstrating that origami could employ Lill's method, a somewhat obscure, 60 Geometric way to solve a cubic. The orientation of Lill's right-angled path has changed here but it can still be traced out on this piece of origami paper, if one takes its dimensions to be 10×10 units square. The points O and D of the 60 Geometric proof are, respectively, the open and filled stars in this proof. The key step #7 consists of making a fold that places these stars on adjacent edges, and the two different ways of doing this correspond to the two solutions.

In *Mathematics Under the Microscope: Notes on Cognitive Aspects of Mathematical Practice*, the University of Manchester mathematics professor Alexandre Borovik presents origami as an example of a "mathematical meme"—an elementary unit of cultural transmission with "the intrinsic property that it increases the precision of reproduction and error correction of the meme complexes it belongs to."[82]

40

Induction

Theorem. *If n is a natural number and $n^3 - 6n^2 + 11n - 6 = 2n - 2$, then $n = 1$ or $n = 4$.*

Proof. Direct calculation shows that the cubic equation is satisfied for $1 \leq n \leq 4$ only if $n = 1$ or $n = 4$. It remains to show that $n^3 - 6n^2 + 11n - 6 \neq 2n - 2$ for all $n \geq 5$. Let $P(n)$ be the proposition $n^3 - 6n^2 + 11n - 6 > 2n - 2$. We will show $P(n)$ by induction on $n \geq 5$.

The base case: For $n = 5$, $n^3 - 6n^2 + 11n - 6 = 24 > 8 = 2n - 2$. Thus $P(5)$ is true.

The inductive step: Observe that

$$
\begin{aligned}
(n+1)^3 - 6(n+1) + 11(n+1) - 6(n+1) &= (n^3 - 6n^2 + 11n - 6) + (3n^2 - 9n + 6) \\
&= (n^3 - 6n^2 + 11n - 6) + 3n(n-3) + 6 \\
&> (n^3 - 6n^2 + 11n - 6) + 6 \quad \text{since } n > 3 \\
&> (2n - 2) + 6 \quad \text{by the inductive hypothesis} \\
&> 2(n+1) - 2
\end{aligned}
$$

and thus $P(n+1)$ is true. Then, by induction, we conclude that $P(n)$ is true for every natural number $n \geq 5$. Hence, $n = 1$ and $n = 4$ are the only solutions, as was to be shown. □

The principle of mathematical induction states that a proposition $P(n)$ that depends on a natural number n is true for all n if $P(1)$ is true and $P(n+1)$ is true whenever $P(n)$ is. This doesn't sound like inductive reasoning of the sort used in the natural sciences—and it isn't—but mathematicians do of course reason from particulars, for example when trying to figure out what to prove with the above principle.

According to Florian Cajori's article on the origin of the name "mathematical induction," seventeenth century English mathematician John Wallis was given a hard time by his contemporaries for introducing the word "induction" to describe both the informal and the formal stages of intellectual work. "[Fermat] blames my demonstration by Induction, and pretends to amend it. . . . I look upon Induction as a very good method of Investigation; as that which doth very often lead us to the easy discovery of a General Rule," Wallis complained, and, of another critic, Ismaël Bullialdus, "he thinks I have not done my invention so much honour as it doth deserve."[83]

What is more honorable, conjecturing or proving Wallis's product?

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot \cdots}$$

NEW SOLUTION TO ANCIENT MATHEMATICAL RIDDLE

By ANTONIO da CELLATICO
Published: March 9, 1539

It all started with the oracle at Delphi.

In order to calm the political strife within the ancient Greek city of Delos, the oracle gave the citizens a geometry puzzle. It goes by the name of the "Delian problem," and mathematicians have been mulling it over ever since.

On December 29, an Italian mathematician from Brescia claimed to have solved a related problem that could turn out to be a breakthrough in the Delian problem.

The Brescian, Zuanne de Tonini da Coi is said to have relied on novel techniques belonging to the field known as "Algebra," though details of his solution have been slow to emerge.

Da Coi's claim appeared in a letter to the Milanese medical doctor and mathematician Hieronimo Cardano.

"It's sensational," said Cardano, a recipient of grants from the Marquis del Vasto. "If it turns out to be right, it could signal a revolution in the field."

Others are less convinced.

"Sometimes we arrive at the solution of an equation without yet being able to justify it," said Niccolò Tartaglia, another mathematician from Brescia not involved in the study. "But before I see a proof, it's my habit to maintain doubt."

Near the end of the last century, Luca Pacioli, the Franciscan friar and scientific collaborator with the great Leonardo da Vinci, struggled with similar problems of Algebra. According to scientists familiar with his treatise on the matter, Pacioli even suggested that some of the so-called "cubic equations" might not have solutions.

Such doubts were shared by the Delians, too. The oracle's instruction was to measure a new altar to the god Apollo in the shape of a cube. One of Plato's favorite forms, the cube is a solid in the shape of a die with six square sides all of which are equal.

"The tricky part is that, according to the oracle's demand, the new altar must equal precisely two times the old altar," explained Dr. Cardano. "You can't simply double the side length since that would result in an altar equal eight times the first." As much happiness as an oversized altar might bring Apollo, it wouldn't satisfy the oracle.

In the elaborate terms of mathematics today, da Coi's problem appears no less baffling than that of the ancients. According to his letter, he finds "the cube and eleven times the side and two equal six times the square and twice the side plus six."

And yet da Coi's equation might just have a solution.

Until this new discovery is fully resolved, research will continue. In any case, it seems likely that the Delian Problem will continue to command at least the attention of mathematicians, if not the citizens of Delos, for many years to come.

This style is modeled on various *New York Times* articles published in the last twenty years[84] and Martin Robbins's satirical blog post "This is a news website article about a scientific paper" hosted by the *Guardian*.[85]

The Delian problem is one of the classical construction problems—a question about numbers that can be constructed by straightedge and compass alone. Cardano, Tartaglia, and others were looking for solutions to cubic equations under the assumption that any cube root could be found or, as we say, "extracted." Such distinctions are frequently lost in journalism causing great annoyance to the mathematically-minded but little harm to the general public.

The names and dates here are based on Nordgaard's article "Sidelights on the Cardan-Tartaglia Controversy."[86] We have an account of the Delian problem from Plutarch. Doubling the cube by straight edge and compass was shown to be impossible by Pierre Wantzel in 1837.

42

Analytic

Theorem. *There exists a real number x such that $x^3-6x^2+11x-6=2x-2$.*

Proof. Let $f:\mathbb{R}\to\mathbb{R}$ denote the function whose value at x is the difference of the left and right sides of the equation:

$$f(x)=x^3-6x^2+11x-6-(2x-2).$$

Define $A\subset\mathbb{R}$ to be the subset of the domain of f that maps to strictly negative real numbers

$$A=\{x : f(x)<0\}.$$

Observe that A is a nonempty and bounded subset of the reals. For example, $0\in A$ since $f(0)=-4$, and 10 is an upper bound of A since $f(10)=486$. According to the completeness axiom of real numbers there exists a real number c that is the least upper bound of A

$$c=\sup A.$$

We will show that $f(c)=0$, and c is therefore a root of the equation. Let $\epsilon>0$. By continuity of f, there exists $\delta>0$ such that $|f(x)-f(c)|<\epsilon$ whenever $|x-c|<\delta$. The first inequality implies that

$$f(x)-\epsilon<f(c)<f(x)+\epsilon.$$

According to the properties of least upper bounds, there exists a real number a in the interval $(c-\delta,c]$ such that $a\in A$. Thus $f(c)<f(a)+\epsilon\leq\epsilon$, since $f(a)<0$. Moreover, there exists a real number b in $[c,c+\delta)$ such that $f(b)\notin A$. Now this implies that $-\epsilon\leq f(b)-\epsilon<f(c)$, since $f(b)\geq 0$. Combining the two inequalities yields

$$-\epsilon<f(c)<\epsilon.$$

As these bounds hold for any $\epsilon>0$, we conclude that $f(c)=0$ as claimed. □

A major branch of mathematics, analysis is the study of limiting processes, including infinite sequences and sums and the derivative and integral of calculus. Analysis is also the site of one of the most recognizable expository styles: the *delta-epsilon style* of proof. In the 1935 article "Variations du style mathématique," Claude Chevalley describes its two-fold characteristics: "First of all, in the intensive, and at times immoderate, usage of the 'ϵ' equipped with various indexes.... Secondly, in the progressive replacement of equality for inequality in the demonstrations as well as in the results."[87]

Though the theorem only asserts the existence of a root c, the proof provides bounds $0 < c < 10$. Sharper bounds appear in 54 Arborescent and 69 Statistical.

The argument here is essentially the proof of Bolzano's theorem, a particular case of the intermediate value theorem, which states that a continuous function attaining positive and negative values must have a root. The theorem is named for the nineteenth century mathematician and philosopher Bernard Bolzano, who proved, using analytic techniques alone, that *every* nonconstant polynomial has at least one *complex* root $a + b\sqrt{-1}$. This more general result is known as the fundamental theorem of algebra. See 51 Topological for an example that illustrates how this theorem can be proved using geometric methods.

FADE IN:

EXT. RENAISSANCE MILAN, ITALY DAY

A town square. Church bells ring the noon hour. A crowd has gathered. CARDANO's protegé, the young LUDOVICO FERRARI, stands at the center, with his ASSISTANT to the side.

FERRARI

Let him who calls himself Niccolò Fontana Tartaglia of Brescia come forth and defend his honor. He who prints libelous attacks upon our esteemed professor Girolamo Cardano in a vain effort to bring disrepute to our University of Milan, the greatest in all Lombardy.

FERRARI looks over the crowd. PAN OVER as the crowd searches itself. PUSH IN to. . .

TOWNSPERSON

Brixians are chickens!

The crowd cheers. Finally, FOLLOW a small man wearing sandals and modest linen robes as he makes his way through the crowd.

TARTAGLIA

Here, here I am.

TOWNSPERSON

Ta-ta-tar-ta-ta-taglia!

The crowd jeers and makes way.

FERRARI
(To TARTAGLIA)

Are you prepared?

TARTAGLIA

I did not traverse one hundred forty-three thousand six hundred and seventy nine braccio from Braccia, no, Brescia to Milan. . .

The crowd laughs. FERRARI gestures to quiet them.

FERRARI

Let him finish!

TARTAGLIA

To give counting lessons to a servant boy.

The crowd "ooh"s.

FERRARI

Indeed, certainly you did not. No more can one count feathers on a boar or trotters on a chicken.

The crowd laughs.

TARTAGLIA

What?!

FERRARI

Take no offense, ’twas only an observation of fact. For the Milanese pace is thrice the Brescian, and thus your one hundred forty-three thousand six hundred and seventy nine braccio is our forty-seven thousand eight hundred and ninety-three brabucco.

The crowd cheers.

TOWNSPERSON

How you like ’em countin’ lessons.

(Laughter)

TARTAGLIA
(To FERRARI)

Where is your master?

FERRARI

If his Excellency Signor Girolamo Cardano suffered every fool to challenge his authority in the art of algebra, he’d never have time for his patients. Surely if your genius overflows the bounds of our physician of Milan, you can best this lowly creature of his.

FERARRI bows. TARTAGLIA nods.

FERRARI
(To his ASSISTANT)

As he commands.

ASSISTANT

I hereby declare that on this the tenth of August in the year fifteen hundred and forty-eight of our Lord in our fair city of Milan, the visitor Niccolò Fontana Tartaglia of Brescia challenges Girolamo Cardano, represented here by Ludovico Ferrari Esquire, to a duel of the mind. The dishonored shall pay the winner two hundred scudi.

FERRARI jingles a leather pouch over head. The crowd cheers.

ASSISTANT
(Continuing)

Both parties have exchanged thirty problems. As visitor, Tartaglia, you may begin by presenting your solution to the first problem, which I shall now read.

He unfurls a scroll.

ASSISTANT
(Reading)

A bankrupt merchant begins for the repayment of half his debt in three years. The agreement is that each year he pays the same proportion of the remainder. He now. . .

TARTAGLIA

What? I've never seen this. . .

FERRARI

Is there a problem? If you wish to drop your challenge, simply say so.

TARTAGLIA

N-n-no! Continue.

ASSISTANT
(Reading)

He now wishes to know the amount of the initial payment he must make in order that after three years he will have repaid half his capital plus a debt fee equal three quarter his first payment.

CLOSE ON TARTAGLIA. He mops his brow with a handkerchief.

TARTAGLIA
(V.O.)

Suppose the debt were two hundred. The first Year's payment. . .

CLOSE ON FERRARI'S PURSE.

TARTAGLIA
(V.O.)

By the second year he will pay it less half its square.

Images of dancing coins float with his mental calculations.

TARTAGLIA
(V.O.)

And the third year he pays the first payment less its square plus a quarter its cube—all of which must come to one hundred plus three quarters of the first. . .

TOWNSPERSON
(Tapping his walking stick on the ground)

Ta-ta-ta-ta. . .

TARTAGLIA sweeps the TOWNSPERSON's walking stick out from under him with his foot. The TOWNSPERSON collapses, much to the amusement of the crowd. Without missing a beat, TARTAGLIA picks up the walking stick and continues his calculation, tracing figures in the dust with its end. TRACK as he writes "cube and 9 cose equal 6 square and 4." The crowd is silent. Then "cube equal 3 cose and 2."

TARTAGLIA'S MOTHER
(V.O. in Italian verse)

In el secondo de cotesti atti Quando che'l cubo restasse lui solo Tu osservarai quest'altri contratti. . .

TARTAGLIA stops writing.

TOWNSPERSON

He's stumped!

TARTAGLIA

(To the TOWNSPERSON)

Half. He repays half his debt in the first year.

ASSISTANT

Correct.

The crowd gasps. FERRARI-CLOSE. Sweating.

FERRARI

(Grabbing the scroll from his ASSISTANT)

Let me see that.

GIROLAMO CARDANO, disguised in monastic robes, peers out from behind a fruit vendor's cart.

TARTAGLIA

(Shouting into the crowd)

Cardano, where are you?

CARDANO leaps back behind the cart, takes out a pen and scribbles on a scrap of paper that he gives to a BOY. Without being noticed, the BOY darts through the crowd and delivers the message to FERRARI. He places it in the scroll, which he returns to the ASSISTANT.

FERRARI

Next question!

ASSISTANT

(Reading the scrap of paper)

Four times a number exceeds by two the product of its square and cube. What is the number?

TARTAGLIA

A qui-qui-quintic?

The crowd laughs.

TARTAGLIA
(to FERARRI)

I know my limitations. Do you yours?

ASSISTANT

What is your answer?

TARTAGLIA

What is yours?

ASSISTANT

Your query is out of turn. Answer or admit ignorance.

TARTAGLIA is silent, but then begins again his calculations in the sand. The crowd steps back to make more room. CARDANO elbows the head of the BOY to get a better view. TARTAGLIA nears the fruit cart, but stops abruptly.

TARTAGLIA

I cannot extract the root.

ASSISTANT

You admit ignorance then?

TARTAGLIA turns his back to FERRARI, and slowly walks away. He hands the stick to the TOWNSPERSON.

TARTAGLIA

Of course.

TOWNSPERSON

The chicken is cooked!

The crowd laughs and cheers.
TARTAGLIA-CLOSE. Brow furrowed.

TARTAGLIA
(V.O.)

At least I was honest in my ignorance. No one, least of all Ferrari, could solve the problem.

REVERSE to FERRARI watching him leave. With TARTAGLIA nearly out of sight, CARDANO steps forward to examine TARTAGLIA's writing in the dust.

CARDANO
(Reading out loud)

One thousand and seventeen parts in two thousand. . .

Pressed by the crowd, FERRARI backs into CARDANO and tramples the solution.

CARDANO
(To FERRARI)

You ignoramus!

TARTAGLIA-CLOSE. The crowd in the distance. He dons a small but victorious smirk.

FADE OUT:

THE END

The hero genius is one of the most well worn narratives in the dramatic depiction of mathematics, closely accompanied by the mad genius. This characterization dates at least as far back as antiquity, when Archimedes, unable to contain his excitement at discovering the principle of displacement in his bath was supposed to have run through the streets of Syracuse in his birthday suit shouting “Eureka! Eureka!” With his discovery, he solved Hiero’s problem, and such challenges are another feature of mathematical dramatizations that persist to today. Surprisingly, the mathematical duel between Tartaglia and Cardano is part of historical record, though the dialogue here is based on cinematic tropes of today.

According to a series of public letters between Tartaglia and Cardano’s protegé Ludovico Ferrari, after repeated entreaties from the latter, Tartaglia divulged his solution to the reduced cubic, which he had committed to memory in the form of a poem (see 81 Doggerel note). Cardano promised not to publish Tartaglia’s solution. Some time later Cardano found evidence that the mathematician Scipione del Ferro, a professor at the University of Bologna, knew the formula prior to and independent of Tartaglia, and Cardano satisfied himself that it wouldn’t be breaking his promise to publish the solution since it wasn’t really Tartaglia’s in the first place. Tartaglia saw it differently. In a series of pamphlets published at his own expense, he waged a war against Cardano and this was the background for the mathematical showdown that took place at ten o’clock in the morning on August 10, 1548 at the Church of Zoccolante in Milan. By all accounts, Cardano didn’t show up to the duel, but his “creature,” as Tartaglia called him, did.

The screenplay format here is based on guidelines published by the Academy of Motion Pictures Arts and Sciences.[88]

44

Omitted with Condescension

There is a simply beautiful theorem which provides all solutions of the equation $x^3 - 6x^2 + 11x - 6 = 2x - 2$. Alas, any further explanation would deny you the satisfaction of discovering it on your own. . . .

The second sentence paraphrases a line from *The Geometry* of René Descartes: "I shall not stop to explain this in more detail, because I should deprive you of the pleasure of mastering it yourself, as well as of the advantage of training your mind by working over it, which is in my opinion the principal benefit to be derived from this science. Because, I find nothing here so difficult that it cannot be worked out by any one at all familiar with ordinary geometry and with algebra, who will consider carefully all that is set forth in this treatise."[89]

This echoes a similar sentiment appearing in a recent translation of Omar Khayyam's book of Algebra: "I could have presented an example for each of the types and kinds [of problems] to prove validity. But I avoided lengthy arguments and I limited myself to general rules counting on the intelligence of the student, because any person who has enough intelligence to understand this article would not fail to produce what he needs of partial examples. God gives guidance to the good, and on Him we depend."[90]

Sherlock Holmes is more to the point: "I am afraid that I rather give myself away when I explain. Results without causes are much more impressive."[91]

45

Verbal

Suppose the real cubic polynomial x cubed minus six x squared plus eleven x minus six equals two x minus two. Or, simply, x cubed minus six x squared plus nine x minus four equals zero. Then x has two roots, one and four. The idea of the proof is to transform the equation by a change of variable into a degree six equation that factors as a perfect square of cubes. This is usually done in two steps. First, get rid of the quadratic term by substituting y plus two in place of x. This results in the so-called depressed cubic, y cubed minus three y minus two. Second, replace y by the sum of z plus one over z. After multiplying the result by z cubed to clear the denominator, we get z to the sixth power minus two z cubed plus one. This factors nicely as z cubed minus one times z cubed minus one. So, z is a cube root of unity: either one, or minus a half plus half the square root of minus three, or minus a half minus half the square root of minus three. If z equals one, then y is two, and x, four. Taking z to be either of the two complex cube roots results in y equal to minus one, which gives the double root, x equal one. So, there are only two distinct roots of the cubic, one and four.

This proof began as a direct transliteration of 18 Indented, which turned out be unbearably monotonous. The total absence of flow and emphasis made it difficult to check. The bilateral structure that symbolic equations impose was completely obscured. To alleviate this, I added punctuation and parallel grammatical structures where possible.

It wasn't until the late Renaissance that one could rely on symbolic algebra when writing a proof. Did proofs like Cardano's (see 7 Found) sound as awkward (his translator uses the word "crude") to ears back then as they do to ours?

I wish there were more math audiobooks. Not only because I have a long commute, but also because maybe then more mathematical writers would develop a "voice."

46

Cute

Theorem. *Let x be real. If $x^3 - 6x^2 + 9x - 4 = 0$, then x equals* 1 *or* 4.

Proof. Suppose x is rational. A rational root of a monic polynomial is a divisor of its constant term. Furthermore, the alternating coefficients imply a positive root. Since 2 is not a solution, another root must be irrational. But irrational roots come in conjugate pairs. $\square$

"Cute" isn't a technical term, its meaning is likely to vary from reader to reader, and, in fact, I'm not even sure how widespread its use is. Nevertheless, this strikes me as cute. It is short, and it seems to get away with something. The supposition that x is rational stops just short of begging the question. Algebra students will recognize this as the *rational root test*, but if you haven't seen it before, it feels a little like cheating.

The defining feature of a cute proof—the short cut—ought to be elementary with respect to the theorem statement. A shortcut relying on some significantly higher-power mathematics is the sign of a proof that is merely 47 Clever.

47

Clever

Theorem. *Let $x \in \mathbb{R}$. If $x^3 - 6x^2 + 11x - 6 = 2x - 2$, then $x = 1$ or $x = 4$.*

Proof. Let $x = 2\cos\theta + 2$. Then the given cubic equation,

$$x^3 - 6x^2 + 9x - 4 = 0$$

becomes

$$4\cos^3\theta - 3\cos\theta - 1 = 0.$$

Note the trigonometric identity:

$$4\cos^3\theta - 3\cos\theta - \cos 3\theta = 0.$$

Hence $\cos 3\theta = 1$, and $x = 2\cos[(\arccos 1)/3] + 2$. $\square$

This could also be called “slick.” The shortcut invokes, out of nowhere, a trigonometry identity. This substitution is due to sixteenth century French mathematician François Viète, who was a very clever fellow indeed.[92] The Spanish King Philip II accused him of resorting to “black magic,” as he could find no other explanation for how, in service to France’s Henry IV, he had cracked a Spanish cipher.[93]

Proofs that borrow from a domain of knowledge perceived to be distant from the context of the theorem are deemed *impure* by proof theorists. They can appear simpler than their purer counterparts, though it’s not clear how precisely to evaluate observations such as this.[94]

48

Computer Assisted

```
Require Import Omega Arith.
Lemma WplusX_ne_YplusZ : forall w x y z:nat,
w > 0 /\ x > 0 /\ y > 0 /\ z > 0 /\ w > y /\ x > z -> w + x <> y + z.
Proof.
  intros; intuition.
Qed.
Lemma XmultY_gt_Z : forall x y z:nat, y >= 1 /\ x >= z /\ y >= z -> x * y >= z.
Proof.
  intros; destruct H; destruct H0.
    rewrite <- mult_1_r; apply mult_le_compat.
  exact H0.
  exact H.
Qed.
Ltac use_XmultY_gt_Z :=
  apply XmultY_gt_Z; intuition.
Theorem The_Cubic : forall x:nat, (x = 1 \/ x = 4 -> x^3 + 9 * x = 6 * x^2 + 4)
/\ (x = 0 \/ x = 2 \/ x = 3 \/ x = 5 \/ x = 6 \/ x > 6 -> x^3 + 9 * x <> 6 * x^2 + 4).
Proof.
  intros; split.
    intros; destruct H.
         rewrite H; easy.
         rewrite H; easy.
    intros; destruct H.
         rewrite H; easy.
      destruct H.
         rewrite H; easy.
      destruct H.
         rewrite H; easy.
      destruct H.
         rewrite H; easy.
      destruct H.
         rewrite H; easy.
    apply WplusX_ne_YplusZ; intuition.
      do 2 use_XmultY_gt_Z.
      do 2 use_XmultY_gt_Z.
      assert (x^3 = x^2 * x) by (simpl; intuition).
         rewrite H0; do 2 rewrite mult_comm.
      assert (6 * x^2 = x^2 * 6) by intuition.
         rewrite H1; apply mult_lt_compat_l; intuition; use_XmultY_gt_Z.
Qed.
```

Like any proof, the text here is both a program for verifying the truth of the theorem (for natural numbers) and a record of the thinking that led to that verification. Here the "thinking" is shared between the human prover and the computer prover, in this case the proof assistant known as Coq—one among many such software systems available today.[95] On the human side, the credit for this proof goes to two undergraduate researchers working on this project, Sarah Dennis and Marshall Pangilinan. The assistant works by applying "tactics" to unpack (`intros`, `destruct`, `split`), rewrite (`rewrite`, `apply`), or otherwise transform the desired conclusion or goal into simpler subgoals. What is considered simple or frustratingly elementary to a human is often complex, and Coq has a range of tactics that one can import so as to automate the search for implicit logical connections (`easy`, `intuition`, `exact`). Alternatively, provers can define their own tactics with the command `Ltac`.

Since Kenneth Appel and Wolfgang Haken's 1976 proof of the four color theorem, much has been written about mathematicians' distaste for computer-checked proofs. The trials, tribulations, and grudging acceptance of Thomas Hales's *A proof of the Kepler conjecture* is a more recent example.[96] For a refreshing viewpoint on this decades-old debate, visit Dr. Z's Opinions where you'll find articles like "Don't Ask: What Can The Computer do for ME?, But Rather: What CAN I do for the COMPUTER?"[97]

49

Outsider

Dear Dr. Ording,

Thank you so much for your thoughtful letter (email dated Jan 18 at 1:26 PM).

There is no non-trivial representation for the “x” in your proposition, which can be stated as $(x-1)^2(x-4)=0$, then $x=1$ or $x=4$. This is because the “x” **is not a variable** in your proposition. Its value(s) can only be logical or unreasonable (no solution). That your proposition has a reasonable solution is credited to logic, but does not reveal the reasonableness of the quantum postulate that created logic in the first place. The work I offer is about that reasonableness. The quantum postulate created logic to constrain instability but leave at liberty pre-established harmony.

I am hoping that if you keep rereading what I sent you last 5 May 2017, each rereading will bring you closer and closer to understanding my work. Knowledge and understanding are not the same thing. In logic they are both subject to variableness. BUT knowledge of the quantum IS understanding. There is no variableness in the quantum.

In my work a 1st Generation is a non-trivial representation of its original argument. This 1st Generation is created by the quantum postulate the moment the last “**variable**” of an original proposition has been replaced with a logical non-trivial representation of that variable. At that instant the original proposition is initialized in the 1st Generation in accordance with the logic used to create the 1st Generation. The quantum postulate does not change any assumptions, only conjectures, i.e., the original conjecture with a non-trivial representation. If a 1st Generation reveals a change in the proposition’s assumption it is an indication that the original proposition is empirically false. A logical analysis of such a 1st Generation will tell you exactly what went wrong, and what can be done to re-establish harmony in the false proposition.

I’m not sure that my quantum work could fit into your book, but I wish you the very best. And if you think it might fit in, I would be happy to work with you.

With best regards,
John P. Colvis

PS: If you wish to share my work with others, an easy way has recently been presented. The Marquis Who’s Who publishers have created a website to help me bring this work to the world. The website, which they maintain, is: http://www.johnpariscolvis.com.

Dear Mr. John Colvis,

Last spring I received your letter on the quantum postulate. Thank you for sending it. Though I can't claim to grasp it fully, I found it very intriguing.

Recently I was rereading it, and it occurred to me that it might be applicable to a project that I've been working on—a manuscript on different styles of mathematical proof. Each brief chapter of the book will present a different way of deducing the roots of one and the same elementary cubic equation. It includes styles inspired by history (antiquity, medieval, modern), subject areas (geometry, probability, topology), tools (computer, calculator, slide rule), and a broad range of sources. I am aiming to reach a math-interested general reader.

The proposition proved by each chapter is:
If x^3-6x^2+11x-6 = 2x-2, then x = 1 or x = 4.

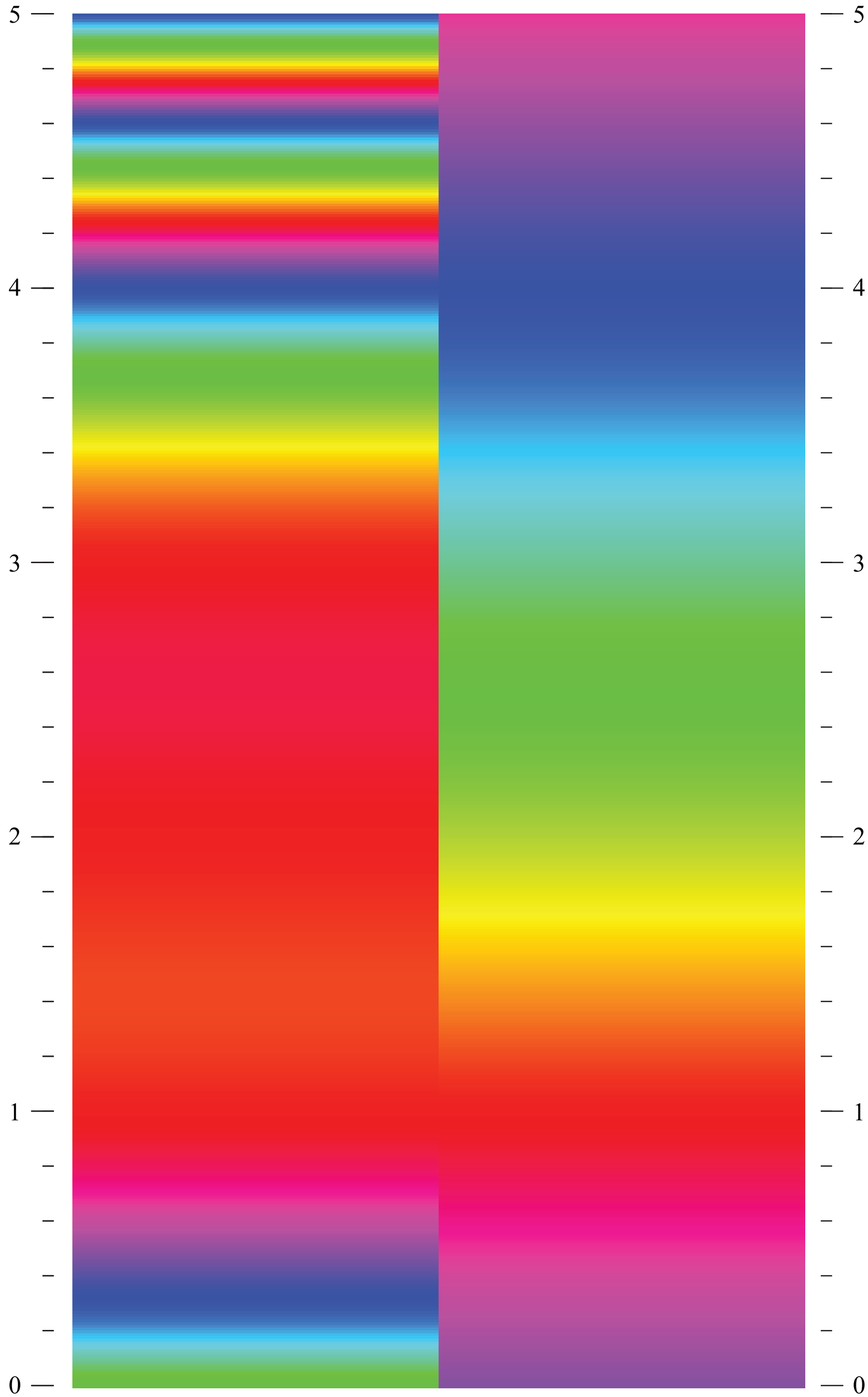
There have been many instances of discovery in math and science that were achieved by individuals working outside the professional community. A NYTimes article once called such discoveries "outsider math" (http://www.nytimes.com/2002/12/15/magazine/the-year-in-ideas-outsider-math.html). In my own bailiwick of knot theory, Kenneth Perko is probably the best known example. He, like you, had formal training but was not an academic at the time he discovered his "Perko pair."

I'm writing to ask if you would possibly be interested in contributing a proof of the above proposition that illustrates your quantum postulate?

Of course your contribution would be fully acknowledged. Each proof in the collection runs about 1-2 pages and is followed by a brief commentary on the pattern of the style and its sources. The manuscript is in the final stages of editing.

If it'd be helpful, I'm happy to send more information about the project and its aims. In any case, thanks again for your letter.

Best regards,
Philip Ording[98]



50

Chromatic

The two spectra represent the two sides of the equation

$$x^3 - 6x^2 + 11x - 6 = 2x - 2.$$

At height x the hue on the left is proportional to $f(x) = x^3 - 6x^2 + 11x - 6$ and the hue on the right is proportional to $g(x) = 2x - 2$. A unit corresponds to a difference of 40° in hue. The red band (0°) at $x = 1$ and the blue band (240°) at $x = 4$ are the two solutions.

This diagram is more typical of "false color" graphics appearing in the natural sciences, where color functions to accentuate and elucidate features of complicated data sets (think of a weather map). Some mathematicians also confront complicated data sets—especially since the advent of electronic computing—but journal articles are less likely to feature color graphics than one might expect.

The nineteenth century mathematician, engineer, and revolutionary Oliver Byrne thought this was an oversight and produced *The First Six Books of the Elements of Euclid in Which Coloured Diagrams and Symbols Are Used Instead of Letters for the Greater Ease of Learners* to prove it. His introduction makes a hard sell, claiming that students would absorb Euclid from his book in one third the time that it takes without color. He even defends against would-be detractors:

> THIS WORK has greater aim than mere illustration; we do not introduce colours for the purpose of entertainment, or to amuse *by certain combinations of tint and form*, but to assist the mind in its researches after truth, to increase the facilities of instruction, and to diffuse permanent knowledge.[99]

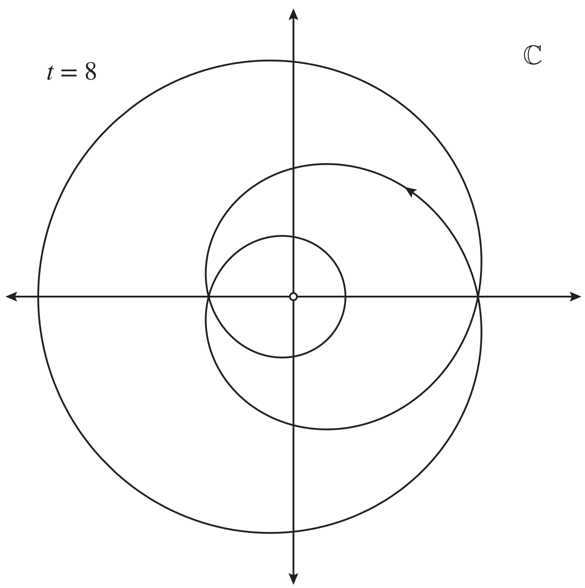
Some people can't help but see mathematics in color. Physicist Richard Feynman reported, "When I see equations, I see the letters in colors—I don't know why. As I'm talking, I see vague pictures of Bessel functions from Jahnke and Emde's book, with light-tan *j*'s, slightly violet-bluish *n*'s, and dark brown *x*'s flying around. And I wonder what the hell it must look like to the students."[100]

51

Topological

Theorem. *There exists a complex number z such that* $z^3 - 6z^2 + 11z - 6 = 2z - 2$. *For every such number,* $|z| \leq 8$.

Proof. Let f be the function $f(z) = z^3 - 6z^2 + 11z - 6 - (2z - 2) = z^3 - 6z^2 + 9z - 4$. Consider the path in the complex plane taken by $f(z)$ as z moves counterclockwise around a circle centered at the origin with radius $t \geq 0$. If $f(z) \neq 0$ for all z in the circle, then we may define the winding number $\omega = \omega(t)$ of f to be the number of times this path wraps around the origin. That is, ω counts the net angle swept out by the line between the origin and $f(z)$, as a multiple of 2π.



When $t = 0$, the circle reduces to a single point at the origin. Since the image $f(0)$ of the origin is the constant -4, this "path" does not wrap around the origin at all, and $\omega(0) = 0$. Now, suppose $t > 8$, then $|f(z) - z^3| = |-6z^2 + 9z + 4| \leq 6|z|^2 + 9|z| + 4$, which equals $t^2\left(6 + \frac{9}{t} + \frac{4}{t^2}\right) < t^2 \cdot 8 < t^3 = |0 - z^3|$. Thus the distance $|f(z) - z^3|$ between f and the cube z^3 is less than the distance from the origin to the cube z^3, so they have the same winding number. Since the winding number of z^3 is 3, we have $\omega(t) = 3$ for $t > 8$.

This implies that $f(z) \neq 0$ for $|z| > 8$, as any such root would lead to $\omega(t) = \omega(|z|)$ being undefined there. Moreover there must exist at least one root z within the disk $|z| \leq 8$. By definition, ω is an integer-valued function that depends continuously on t, as f is a continuous function of z. Since the only continuous integer-valued functions are constants, the absence of a root in $|z| \leq 8$ would imply that ω were constant, contradicting the fact that it takes the values 0 and 3. □

Topological

The proof is modeled on a topological proof that every nonconstant polynomial has a root in the complex plane, the so-called fundamental theorem of algebra.[101] Like analysis, topology deals with notions of continuity, but in a more generally conceived setting than the real or, for that matter, complex numbers. A memorable spoof of different mathematical styles of reasoning is supplied by H. Pétard's 1938 "A Contribution to the Mathematical Theory of Big Game Hunting," in which one finds:

> 7. A TOPOLOGICAL METHOD. We observe that a lion has at least the connectivity of the torus. We transport the desert into four-space. It is then possible to carry out such a deformation that the lion can be returned to three-space in a knotted condition. He is then helpless.[102]

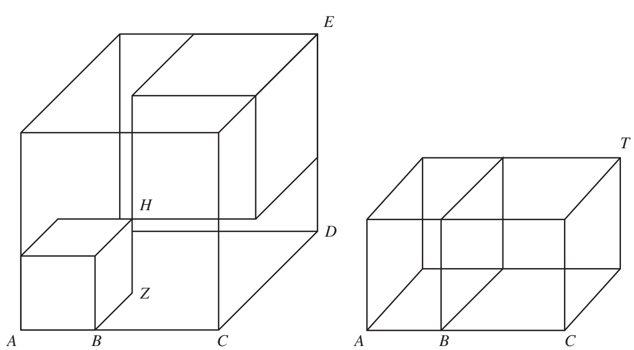
PROPOSITION

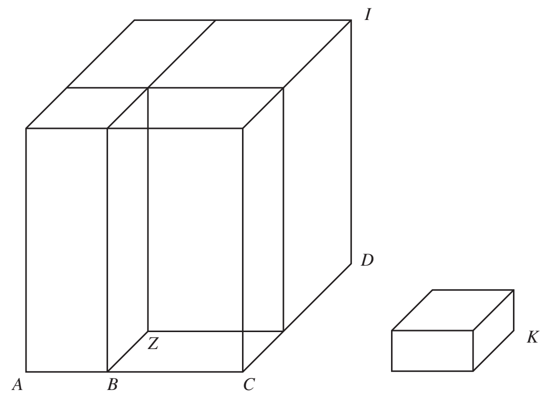
If the cube on a straight line and the parallelepipedal solid with base nine and height the straight line are equal to the parallelepipedal solid with the square on the straight line as base and height six and the parallelepipedal solid with base four and unit height, the straight line has magnitude equal four units.

Let AC be the straight line, AD the square on AC, AE the cube on AC, AT the parallelepipedal solid with base nine and height AC, AI the parallelepipedal solid with base AD and height six, and K the parallelepipedal solid with base four and unit height, such that AE and AT equal AI and K.

I say that AC has magnitude four.

Let AC be cut at B such that three times AB coincides with DI. Describe the square AZ on AB. Complete the rectangles ZD and ZC. Let the solids AE, AT, and AI be cut by a plane carried through B and perpendicular to AC. Describe the cube AH on AB, and complete the parallelepipedal solids HE, CH, and DH. The solid HE coincides with the cube on BC, and the sum of CH and DH coincides with the solid contained by AC, AB, and BC.





If a straight line is cut at random, then the square on the whole equals the sum of the squares on the segments plus twice the rectangle contained by the segments. Hence *AD* equals the sum of *AZ*, *ZD*, and twice *ZC*. Parallelepipedal solids which are of the same height are to one another as their bases. Hence *AI* equals the sum of one each of the parallelepipedal solids with height six and base *AZ* and base *ZD* plus twice the parallelepipedal solid with height six and base *ZC*.

If a parallelepipedal solid is cut by a plane parallel to the base, then the height is to the height as the solid is to the solid. Therefore *AT* equals nine times *AB* and nine times *BC*.

If a straight line is cut at random, then the cube on the whole equals the sum of the cubes on the segments plus three times the parallelepipedal solid contained by the whole and the segments. Therefore *AE* equals the sum of the cube on *AB*, the cube on *BC*, and three times the solid contained by *AC*, *AB*, and *BC*. Things which coincide with one another are equal, and a multiple of a sum of magnitudes is equal to the sum of the magnitudes each multiplied by the same multiple, thus *AE* equals the sum of *AH*, *HE*, three times *DH*, and three times *CH*.

By assumption, AE and AT equal AI and K. Hence the sum of AH, HE, three times DH, three times CH, nine times AB, and nine times BC equals the sum of one each of the parallelepipedal solids with height six and base AZ and base ZD plus twice the parallelepipedal solid with height six and base ZC and K. If equals are subtracted from equals, the remainders are equal. Hence HE equals the sum of one-third BT and one half K.

But HE coincides with the cube on BC, and one-third AT coincides with the parallelepipedal solid of base three and height BC, hence they are, respectively, equal. Things which equal the same things also equal one another, therefore the cube on BC equals the parallelepipedal solid with base three and height BC and half K.

If the cube on a straight line is equal to the parallelepipedal solid with base three and height the straight line and the parallelepipedal solid with base two and unit height, then the straight line has magnitude equal two units. Thus, BC has magnitude equal two units. Since AB is also two units, AC is thus four units.

Therefore, if the cube on a straight line and the parallelepipedal solid with base nine and height the straight line are equal to the parallelepipedal solid with the square on the straight line as base and height six and the parallelepipedal solid with base four and unit height, the straight line is four units. The very thing it was required to show.

In the Japanese language, numbers cannot be used by themselves to count objects as in the English phrases “two sheets of paper” and “two pencils.” One must attach the appropriate counter word with the number. For thin flat objects like sheets of paper the counter is 枚 (*mai*), for long thin objects like pencils the counter is 本 (*pon*).

In Euclid’s *Elements*, you can’t compare a square to a cube unless you find a representation for each in terms of the same dimension. After dispensing with anachronistic negative signs, the cubic equation asserts that the volume x^3+9x is congruent to the volume $6x^2+4$.

Book XI of the *Elements* concerns solid geometry, and it served as the model for this proof.[103] *A priori*, the geometric constraint here does not rule out the case $AC=1$, but this proof does.

Apparently most of the theorems in the *Elements* are not original to Euclid, but the logical presentation of the axioms, constructions, propositions, and proofs may be. Compared to the Socratic dialogues of just a century prior, the “deductivist style,” as Lakatos’s called it,[104] sounds authoritarian and modern.

Once an architect at an interdisciplinary conference hosted by the University of Pennsylvania School of Design confided in me that after studying Euclid for a year he almost went off the deep end. I wasn’t sure what he meant at the time, but I have a better idea now.

Proposition

53

Marginalia

Enunciation
πρότασις

If the cube on a straight line and the parallelepipedal solid with base nine and height the straight line are equal to the parallelepipedal solid with the square on the straight line as base and height six and the parallelepipedal solid with base four and unit height, the straight line has magnitude equal four units.

The proposition is stated in geometric terms that are equivalent to the solution $x=4$ of the cubic equation $x^3+9x=6x^2+4$.

Setting out
ἔκθεσις

Let AC be the straight line, AD the square on AC, AE the cube on AC, AT the parallelepipedal solid with base nine and height AC, AI the parallelepipedal solid with base AD and height six, and K the parallelepipedal solid with base four and unit height, such that AE and AT equal AI and K.

$AC=x$
$AT=9x$
$AD=x^2$
$AI=6x^2$
$AE=x^3$
$K=4$

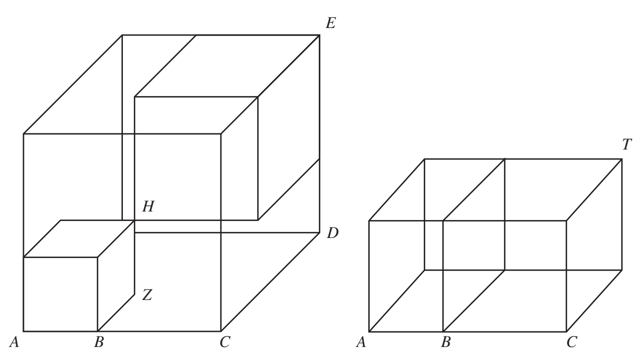
Specification
διορισμός

I say that AC has magnitude four.

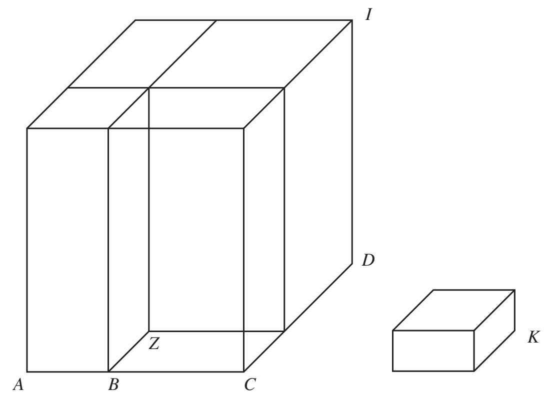
Construction
κατασκευή

Let AC be cut at B such that three times AB coincides with DI. Describe the square AZ on AB. Complete the rectangles ZD and ZC. Let the solids AE, AT, and AI be cut by a plane carried through B and perpendicular to AC. Describe the cube AH on AB, and complete the parallelepipedal solids HE, CH, and DH. The solid HE coincides with the cube on BC, and the sum of CH and DH coincides with the solid contained by AC, AB, and BC.

Thus $AB=2$.
$AZ=AB^2=4$
$ZD=(x-2)^2$
$ZC=2(x-2)$
$AH=AB^3=8$
$HE=(x-2)^3$
$CH=4(x-2)$
$DH=2(x-2)^2$
$CH+DH=2x(x-2)$



Codex Vaticanus 532 has G in place of C.



Proof *ἀπόδειξις* If a straight line is cut at random, then the square on the whole equals the sum of the squares on the

II.4 segments plus twice the rectangle contained by the

$(a+b)^2 = a^2+b^2+2ab$ segments. Hence AD equals the sum of AZ, ZD, and twice ZC. Parallelepipedal solids which are of the $AD = AZ + ZD + 2ZC$

XI.32 same height are to one another as their bases. Hence AI equals the sum of one each of the parallelepipedal solids with height six and base AZ and base ZD plus twice the parallelepipedal solid with height six and base ZC. $AI = 6AZ + 6ZD + 2\cdot 6ZC$ $ZC = 2(x-2)$ $6x^2 = 24+24(x-2)+6(x-2)^2$

XI.40 If a parallelepipedal solid is cut by a plane parallel to the base, then the height is to the height as the solid is to the solid. Therefore AT equals nine times AB and nine times BC. $9x = 18+9(x-2)$

XI.43 If a straight line is cut at random, then the cube on the

$(a+b)^3 = a^3+b^3+3ab(a+b)$ whole equals the sum of the cubes on the segments plus three times the parallelepipedal solid contained by the whole and the segments. Therefore AE equals

CN.4 the sum of the cube on AB, the cube on BC, and three $AE = AB^3 + BC^3 + 3AC\cdot AB\cdot BC$

V.1 times the solid contained by AC, AB, and BC. Things which coincide with one another are equal, and a multiple of a sum of magnitudes is equal to the sum of the magnitudes each multiplied by the same multiple, thus AE equals the sum of AH, HE, three times DH, and three times CH. $AE = AH + HE + 3DH + 3CH$ $x^3 = 8+(x-2)^3+6(x-2)^2+12(x-2)$

By assumption, AE and AT equal AI and K. Hence the sum of AH, HE, three times DH, three times CH, nine times AB, and nine times BC equals the sum of one each of the parallelepipedal solids with height six and base AZ and base ZD plus twice the parallelepipedal solid with height six and base ZC and K.

CN.3 If equals are subtracted from equals, the remainders are equal. Hence HE equals the sum of one-third BT and one half K.

$AE+AT=AI+K$

$AH+HE+3DH+3CH+9AB+9BC=6AZ+6ZD+2\cdot 6ZC+K$

$8+(x-2)^3+6(x-2)^2+12(x-2)+18+9(x-2)=24+24(x-2)+6(x-2)^2+4$

$HE=\frac{1}{3}BT+\frac{1}{2}K$

But HE coincides with the cube on BC, and one-third BT coincides with the parallelepipedal solid of base three and height BC, hence they are, respectively, equal. Things which equal the same things also equal one another, therefore the cube on BC equals the parallelepipedal solid with base three and height BC and half K.

CN.4

CN.1

$HE=BC^3=(x-2)^3$

$\frac{1}{3}BT=3BC$

$BC^3=3BC+\frac{1}{2}K$

Depressed cubic $(x-2)^3=3(x-2)+2$

XI.50 If $y^3=3y+2$ then $y=2$.

If the cube on a straight line is equal to the parallelepipedal solid with base three and height the straight line and the parallelepipedal solid with base two and unit height, then the straight line has magnitude equal two units. Thus, BC has magnitude equal two units. Since AB is also two units, AC is thus four units.

Since K has base four, half K coincides with the parallelepipedal solid with base two and unit height [XI.32], and by CN.4, they are equal.

Conclusion *συμπέρασμα*

Therefore, if the cube on a straight line and the parallelepipedal solid with base nine and height the straight line are equal to the parallelepipedal solid with the square on the straight line as base and height six and the parallelepipedal solid with base four and unit height, the straight line is four units. The very thing it was required to show.

To keep one's wits, as when reading any mathematical text, it's useful to maintain a written commentary nearby. This style could be called *scholia*. For an example of an annotated version of the *Elements*, see the digitized manuscript MS D'Orville 301.[105]

The left margin of this proof includes citations to propositions and common notions of the *Elements*, whether actual (e.g., II.4) or fictitious (e.g., XI.40), as well as the stylistic terms designating the formal divisions of a proposition. Heath discusses these terms—enunciation, setting out, specification, etc.—in the introduction to his translation.[106] Note that the fictitious XI.50 is supposed to solve the depressed cubic $y^3 = 3y + 2$. As an undergraduate researcher working on this project, Xueyi Bu pointed out that this could also be treated volumetrically by adding to each side of the equation a box of unit height and base y^2. This is the route taken in 81 Doggerel.

In the right margin, I have translated the geometry into algebraic equations without which I could not keep track of the proof. *Codex Vaticanus 532* is another fiction inspired by the comments one finds in translations that synthesize multiple editions of a text.

54

Arborescent

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$6<x$}
\AxiomC{$0<6$}
\AxiomC{}
\UnaryInfC{$6<x$}
\BinaryInfC{$0<x$}
\BinaryInfC{$6\cdot x<x\cdot x$}
\AxiomC{$0<6$}
\AxiomC{}
\UnaryInfC{$6<x$}
\BinaryInfC{$0<x$}
\BinaryInfC{$6x\cdot x<x^2\cdot x$}
\UnaryInfC{$6x^2+2x<x^2\cdot x+2x$}
\UnaryInfC{$6x^2+2x<x^3+2x$}
\UnaryInfC{$6x^2+2x+6<x^3+2x+6$}
\AxiomC{}
\UnaryInfC{$6<x$}
\BinaryInfC{$6x^2+2x+6<x^3+2x+x$}
\UnaryInfC{$6x^2+2x+6<x^3+3x$}
\AxiomC{$0<2$}
\AxiomC{$8\cdot 0=0$}
\AxiomC{$0<8$}
\AxiomC{$0<6$}
\AxiomC{}
\UnaryInfC{$6<x$}
\BinaryInfC{$0<x$}
\BinaryInfC{$8\cdot 0<8\cdot x$}
\BinaryInfC{$0<8\cdot x$}
\BinaryInfC{$0<8\cdot x+2$}
\UnaryInfC{$x^3+3x<x^3+3x+8x+2$}
\UnaryInfC{$x^3+3x<x^3+11x+2$}
\BinaryInfC{$6x^2+2x+6<x^3+11x+2$}
\UnaryInfC{$2x+6<x^3-6x^2+11x+2$}
\UnaryInfC{$2x<x^3-6x^2+11x-6+2$}
\UnaryInfC{$2x-2<x^3-6x^2+11x-6$}
\UnaryInfC{$2x-2\neq x^3-6x^2+11x-6$}
\UnaryInfC{$x^3-6x^2+11x-6\neq 2x-2$}
\end{prooftree}
$$

Direct computation for $x=0,2,3,5,6$ yields the same conclusion.
Hence $x\in\mathbb{N}$, $x^3-6x^2+11x-6=2x-2$ implies $x=1$ or $x=4$.

This proof is meant to be read from top to bottom, from “leaf” to “trunk.” Each horizontal line represents a deduction with the statement beneath following from the premise(s) above. If there are two premises over the same line they are conjoined by an invisible “and.” The empty premise above the inequality $6 < x$ indicates that it is the assumption of the proof. The expression $x \in \mathbb{N}$ in the final line is the hypothesis that x is an element of the set of natural numbers. This variation is modeled on the *Gentzen-style* proof calculus of natural deduction, named for the German logician Gerhard Gentzen.[107]

55

Prefix

Dem $\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow = - + - \uparrow x\,3 \times 6 \uparrow x\,2 \times 11\,x\,6 - \times 2\,x\,2 = - + - \uparrow x\,3 \times 6 \uparrow x\,2 \times 9\,x\,4\,0 = - + + - \uparrow x\,3 \times 6 \uparrow x\,2 \times 5\,x \times 4\,x\,4\,0 = + \times - \uparrow x\,2 \times 5\,x - x\,1 \times 4 - x\,1\,0 = \times + - \uparrow x\,2 \times 5\,x\,4 - x\,1\,0 = \times \times - x\,4 - x\,1 - x\,1\,0 \vee = - x\,1\,0 = - x\,4\,0 \vee = x\,1 = x\,4 \rightarrow = - + - \uparrow x\,3 \times 6 \uparrow x\,2 \times 11\,x\,6 - \times 2\,x\,2 \vee = x\,1 = x\,4$

Prefix

The arithmetic operation symbols $+$, $-$, $\times$, $\div$ usually appear between the numbers on which they operate, as in $2+3$ or $x-4$. One downside of so-called *infix* notation is that a sequence of operations can be ambiguous without specifying some convention (that is, an "order of operations") or inserting parentheses. For example $(4-3)-1 \neq 4-(3-1)$.

Polish logician Jan Łukasiewicz avoided parentheses using *prefix* notation—also called *Polish notation* in his honor—which moves the operator symbol from the middle of the two operands to the front of the expression. In this form, the compound subtractions above appear as $- - 4\,3\,1$ and $-4 - 3\,1$, respectively. When we write a function such as $f(x, y) = x - y$, we are using a form of prefix notation.

This convenience was designed to simplify sentential logic, and the prefix notations for "a is equal to b," "ϕ implies ψ," and "P is a demonstration (or proof) of Q" are $= a\,b$, $\rightarrow \phi\,\psi$, and Dem $P\,Q$. The proof follows the logic of 4 Elementary. See also 56 Postfix.

56

Postfix

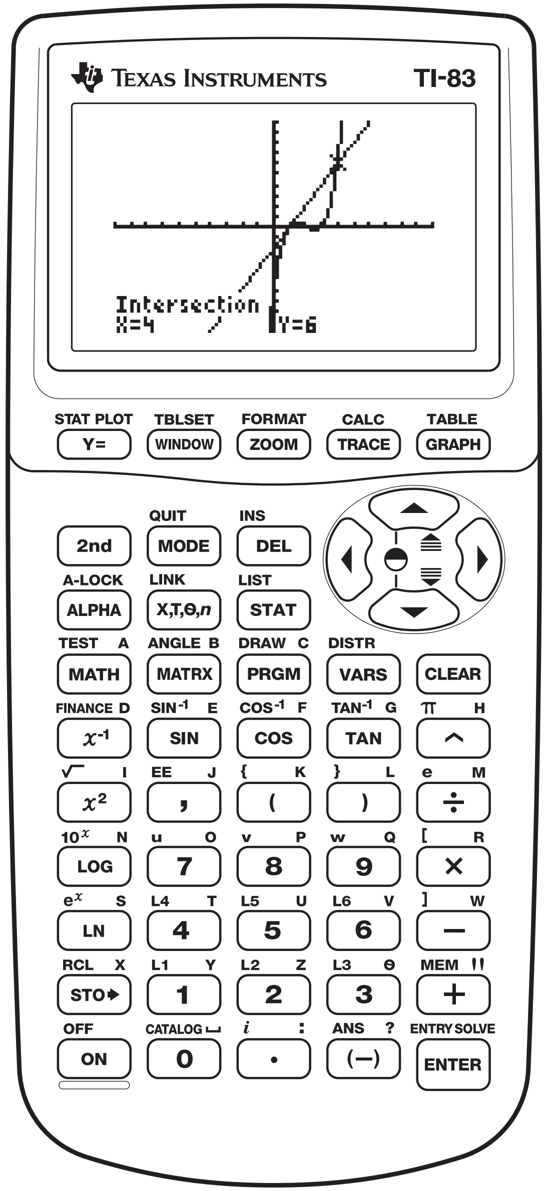
$x\,3\uparrow 6\,x\,2\uparrow\times - 11\,x\times + 6 - 2\,x\times 2 - = x\,3\uparrow 6\,x\,2\uparrow\times - 9\,x\times + 4 - 0 = \rightarrow x\,3\uparrow 6\,x\,2$
$\uparrow\times - 5\,x\times + 4\,x\times + 4 - 0 = \rightarrow x\,2\uparrow 5\,x\times - x\,1 - \times 4\,x\,1 - \times + 0 = \rightarrow x\,2\uparrow 5\,x\times - 4 +$
$x\,1 - \times 0 = \rightarrow x\,4 - x\,1 - \times x\,1 - \times 0 = \rightarrow x\,1 - 0 = x\,4 - 0 = \vee \rightarrow x\,1 = x\,4 = \vee \rightarrow x\,3$
$\uparrow 6\,x\,2\uparrow\times - 11\,x\times + 6 - 2\,x\times 2 - = x\,1 = x\,4 = \vee \rightarrow$ Dem

Compared to 55 Prefix, the operator symbols in this exercise appear immediately following the objects they act on. Also known as *Reverse Polish Notation* or *RPN*, it's the default mode of some electronic calculators. Purists favor RPN calculators for their efficiency. Renouncing parentheses is pretty cool, but the strange satisfaction of pushing and popping elements to and from the waiting list of inputs (i.e., the *stack*) must also account for their appeal. The downside is pretty obvious—can you find the typo? I've corrected the ones I last found, but there always seem to be more.

Solve $x^3 - 6x^2 + 11x - 6 = 2x - 2$.

1. Press **Y=** to display the function editor.
2. Press **X ^ 3 – 6 X x^2 + 11 X – 6**.
3. Use ▼ to enter the second function.
4. Press **2 X – 2**.
5. Press **GRAPH** .
6. Select **intersect** from the **CALC** menu.
7. Use ▼ or ▲ to move the cursor to the first function and press **ENTER** .
8. Use ▼ or ▲ to move the cursor to the second function and press **ENTER** .
9. Press ▶ or ◀ to move the cursor to the point that is your guess as to location of the intersection, and then press **ENTER** .

   The result cursor is on the solution and the coordinate values (4, 6) are displayed.



With the advent of the microchip, handheld electronic calculators became commercially viable in the 1970s. The new technology was inspired by the power of the mainframe computer and the portability of its predecessor, the 75 Slide-Rule. One would think that the ubiquity of smart phones would have made the pocket calculator as obsolete as the slide rule.

But that hasn't happened. Ironically, school teachers and standardized testing companies that require electronic calculators prefer them *because* of their limitations. Calculators can't make phone calls, browse the web, text-message, or take selfies. Texas Instruments, the multibillion-dollar company that also brought us laser-guided bombs, has dominated the educational calculator market for graphing calculators.[108] This is their TI-83, which cost about $100 when it first appeared in 1996.

That these calculators have improved little over the last two decades while maintaining their price point doesn't mean that they're not a good source of distraction, especially with their programming capabilities. I remember my AP Calculus teacher's initial pride turned to exasperation when my classmates figured out how to program the Sierpiński triangle—we couldn't take our eyes off it. More ambitious users have hacked into the operating system, precipitating the "Texas Instruments signing key controversy."[109]

The image of the calculator is used with permission by Texas Instruments, Inc. The idea for this variation came from Robert Dawson of Saint Mary's University, Halifax.

Theorem. *If x is a real number and $x^3 - 6x^2 + 11x - 6 = 2x - 2$, then $x = 1$ or $x = 4$.*

58

Inventor's Paradox

Proof. Solutions to the equation are roots of the polynomial $x^3 - 6x^2 + 9x - 4$. But note that this is the resolvent cubic of the quartic polynomial

$$y^4 + 3y^3 + 6y^2 + 7y + 3.$$

The four roots of a quartic are related to the three roots of its resolvent cubic as follows:

$$\begin{aligned} x_1 &= y_1 y_2 + y_3 y_4 \\ x_2 &= y_1 y_3 + y_2 y_4 \\ x_3 &= y_1 y_4 + y_2 y_3. \end{aligned}$$

Since the quartic factors as $(y+1)^2(y^2+y+3)$, we may, without loss of generality, set $y_1 = y_2 = -1$, $y_3 = \frac{-1+\sqrt{-11}}{2}$, and $y_4 = \frac{-1-\sqrt{-11}}{2}$. Thus, the roots of the cubic polynomial are $x_1 = 4$, $x_2 = 1$, and $x_3 = 1$. □

Inventor's Paradox

"The more ambitious plan may have more chances of success." This is the Inventor's Paradox as Georg Pólya defined it in his "Dictionary of Heuristic."[110] Here, the degree three equation is solved by viewing it from the perspective of a degree four equation. Just as a cubic can be solved by reducing it to a "resolvent" quadratic, a quartic polynomial of the form $y^4 + py^3 + qy^2 + ry + s$ can be resolved to a cubic polynomial of the form $x^3 + bx^2 + cx + d$, where the coefficients are related as

$$b = -q$$
$$c = pr - 4s$$
$$d = 4qs - r^2 - sp^2.$$

To be clear, there's no reason that the degree four equation should be any simpler to solve, whether by factoring or another method, than the cubic. But the exercise is illustrative of the manipulations that often do crack otherwise intractable problems. Cardano's method, for example, is something of an Inventor's Paradox because it solves the cubic by first *doubling* its degree; see 25 Open Collaborative. When is it likely to work? "The more ambitious plan may have more chances of success provided it is not based on mere pretension but on some vision of the things beyond those immediately present," Pólya advises. Nevertheless, even mere pretension sometimes offers a useful glimpse of things beyond those immediately present.

Date: Aug. 03, 2011
Inventor: W. Hedden, Yonkers, N.Y., U.S.A.
Assignee: OBM, Inc., Yonkers, N.Y., U.S.A.
Notice: Subject to any disclaimer, the term of this patent is extended or adjusted under 35 U.S.C. 154(b) by 0 days.
Appl. No.: 20/110,728
Filed: Oct. 15, 2010
Related U.S. Application Data: Continuation of application No. 19/222,025, filed on Jan. 12, 2009, now Pat. No. 18,230,110.
Int Cl.: G06F 7/38
U.S. Cl.: 708/446
References Cited: U.S. Patent 6,823,352 B2, 11/2004 Walster et al., 708/446
Primary Examiner—D. Rasmussen
Attorney, Agent, or Firm—Zoltan LLP

TRADEMARKS. OBM® is a registered trademark of Ozsvath Business Machines Corporation, Yonkers, N.Y., U.S.A.

FIELD OF INVENTION. The present invention disclosure relates to a process of root finding within a computer system. More particularly, this disclosure relates to the method and apparatus for using a computer system to solve a degree three polynomial equation.

ABSTRACT. A computer-readable storage medium storing instructions that, when executed by a computer, cause the computer system to solve the cubic equation $x^3 - 6x^2 + 11x - 6 = 2x - 2$. The method exploits a repeated root and applies the Euclidean algorithm to the associated cubic polynomial and its derivative, finding the roots to be x=1 and x=4. The method is particularly efficient in finding the cubic roots when compared to alternative root finding strategies employed by existing computer systems.

CLAIMS. What is claimed is:

1. A computer-readable storage medium storing instructions that when executed by a computer cause the computer to perform a method for using a computer system to solve the equation $x^3 - 6x^2 + 11x - 6 = 2x - 2$ by using differentiation, the method, as shown in Fig. 1, comprising:
   receiving the cubic equation within the computer system;
   symbolically subtracting the right side B of the cubic from the left side A of the cubic;
   converting the equation into a cubic polynomial $P = A - B$ in standard form;
   differentiating P with respect to x;
   setting Q equal to the derivative P' of P;
   querying the truth value of the equation Q=0;

2. The computer-readable storage medium referenced in claim 1, wherein it further comprises:
   in case $Q = 0$ is false, finding the remainder R upon dividing P by Q;
   setting P equal to Q;
   setting Q equal to R;
   repeating the query at the end of claim 1;

3. The computer-readable storage medium referenced in claim 1, wherein it further comprises:
   in case $Q = 0$ is true, solving the linear equation $P = 0$ for x;
   setting x_1 equal to this solution;
   solving the linear equation $(A - B)/(x - x_1)^2 = 0$ for x;
   setting x_2 equal to this solution;
   outputting the solution set $[x_1, x_2] = [1,4]$;

4. The computer-readable storage medium of claim 1, wherein the method is performed as part of an optimization process.

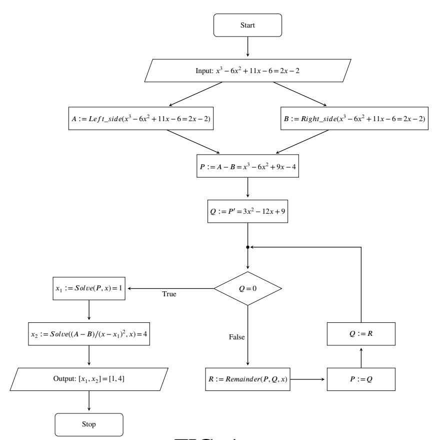


FIG. 1

Patented

Despite the current legal consensus that mathematics can't be patented, mathematics is at the heart of many invention disclosures. U.S. Patent 6,285,999 *Method for Node Ranking in a Linked Database* is an especially influential example that you can easily look up on the Internet, thanks to an application of the invention thus patented. In addition to the class 708/446 *Solving equation* to which this patent belongs, the current classification by the US Patent and Trademark Organization contains many other mathematical sounding classes, including 707/380 *Cryptography*, 708/250 *Random number generation*, 708/443 *Differentiation*, and 708/492 *Galois field*.[111]

The boundaries separating a mathematical formula from an algorithm that expresses a formula from a digital circuit design to implement an algorithm are not sharply drawn. Patentability claims around mathematics return to the Supreme Court as technology advances. The court's opinion in the 1939 case of *Mackay Radio & Telegraph Co. v. Radio Corp. of America* 306 U.S. 86 is typical: "While a scientific truth, or the mathematical expression of it, is not patentable invention, a novel and useful structure created with the aid of knowledge of scientific truth may be."

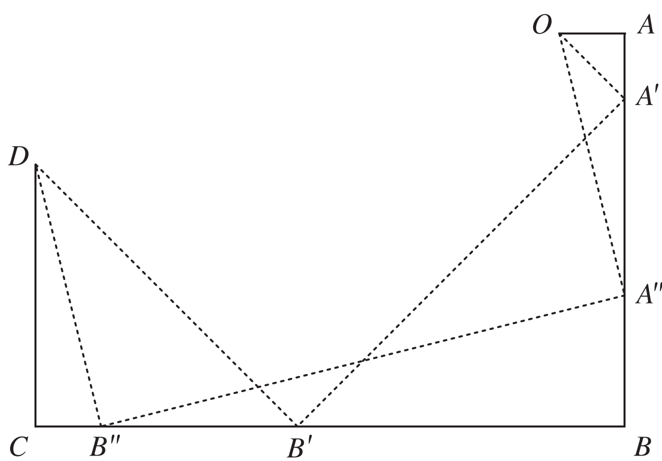
The form and language of this style is modeled on U.S. Patent 6,823,352 B2 Walster et al., Solving a Nonlinear Equation Through Intervalic Arithmetic and Term Consistency.[112]

Theorem. *If x is a real number and $x^3 - 6x^2 + 11x - 6 = 2x - 2$, then $x = 1$ or $x = 4$.*

Proof. We obtain the roots of the equivalent cubic equation $x^3 - 6x^2 + 9x - 4 = 0$ by construction.

From a point O, taken arbitrarily, set out a length OA to the right of O and with magnitude of 1, the coefficient of x^3. Perpendicular to OA, place a line segment AB that extends down from A and has length 6, the magnitude of the coefficient of x^2. Perpendicular to AB, place a line segment BC that extends to the left of B and has length 9, the magnitude of the coefficient of x. Complete the path with a final perpendicular CD that extends up from C and has length 4, the magnitude of the constant term.

Thus we have constructed a right-angled polygonal path $OABCD$ of four sides from O to D. This done, if we can pass from point O to point D by another right-angled polygonal path $OA'B'D$ of only three sides, the intermediate vertices of which A', B' belong respectively to sides AB and BC of the first path, then the number that expresses the length AA' is a root of the equation.



To see that $x = AA'$ satisfies the equation, first note that triangles $\triangle OAA'$ and $\triangle A'BB'$ are similar because their corresponding angles are equal. This implies that $OA : AA' = A'B : BB'$, or, in terms of x, $\dfrac{1}{x} = \dfrac{6-x}{BB'}$, from which we obtain $BB' = x(6-x)$ and $B'C = 9 - x(6-x)$. But triangle $\triangle B'CD$ is similar to $\triangle A'BB'$, and hence also similar to $\triangle OAA'$. This implies $\dfrac{1}{x} = \dfrac{9 - x(6-x)}{4}$, from which the given polynomial is easily recovered.

One can go from O to D by two right-angled polygonal paths $OA'B'D$ and $OA''B''D$. The two lines AA', AA'' represent the two roots of the equation. These lines are respectively equal to OA and quadruple OA; and since they are situated to the right of OA, the roots are positive. Hence the roots are, as claimed, 1 and 4. $\square$

A nineteenth century Austrian engineer named Eduard Lill devised this graphical technique for solving not only cubic equations, but also polynomials of arbitrary degree.[113] Although Lill does not explain how to find the needed right angled paths, once found (e.g., by trial and error), it's not difficult to see that they provide the desired roots. Multiplying the last equation

$$\frac{1}{x} = \frac{9 - x(6 - x)}{4}$$

by $4x$ to clear the denominators produces the equation

$$4 = x(9 - x(6 - x)).$$

This is, essentially, the so-called *Horner's form* of our cubic, named for British mathematician William Horner. Horner's *method* is a quick way to evaluate a polynomial $p(x)$ at $x = x_0$ whereby $p(x_0)$ is obtained as the result of an iterative process of alternately adding the coefficients of the polynomial and multiplying the result by x_0, starting from the leading coefficient. Lill's method, then, is a graphical representation of Horner's method in which each sum is represented by setting out the base of a right triangle and each product is achieved by the formation of a similar triangle.

I wonder if Queneau felt his bearings as a writer and his grasp on language give way as his *Exercises in Style* accumulated. A couple years into this project I realized that I didn't really understand what a polynomial is. Now I'm less sure about addition and multiplication.

61

Modern

Theorem. *Given* $p = (X^3 - 6X^2 + 11X - 6) - (2X - 2)$, *let* $I \subset \mathbb{R}[X]$ *denote the principal ideal generated by* p. *If* $X - x + I$ *is a zero divisor in the quotient ring* $\mathbb{R}[X]/I$, *then* $x = 1$ *or* $x = 4$.

Proof. Observe $((X-1)+I))((X-1)+I)((X-4)+I) = 0$ in $\mathbb{R}[X]/I$, since

$$(X-1)(X-1)(X-4) = X^3 - 6X^2 + 9X - 4 = p.$$

Moreover, $\deg(X-1) = \deg(X-4) = 1$ is less than the degree of the cubic p, and it follows that $(X-1) \notin I$ and $(X-4) \notin I$. Thus $X - 1 + I$ and $X - 4 + I$ are zero divisors in $\mathbb{R}[X]/I$. These are the only linear zero divisors in $\mathbb{R}[X]/I$ as $\mathbb{R}[X]$ is a unique factorization domain. □

The two courses at the heart of an American undergraduate mathematics curriculum are sometimes called modern (or abstract) algebra and modern analysis (see 42 Analytic). Historically, the mathematics they present dates to the nineteenth century, and these courses approach their subject matter from the same high level of abstraction that emerged in that era, one which produced nowhere differentiable continuous functions, non-Euclidean geometry, symbolic logic, and set theory, among other advances. These developments seem to have brought about a consciousness of style itself. One of the most influential graduate textbooks of mathematics in the twentieth century, *Moderne Algebra* by B. L. van der Waerden and based on lectures by Emil Artin and Emmy Noether, is still praised for its expository qualities.[114] "Its simple but austere style set the pattern for mathematical texts in other subjects," Saunders Mac Lane, a leader of twentieth century American mathematics, wrote. "[It] presented… abstractly but without pedantry… a decisive example of a clear and perspicuous presentation."[115]

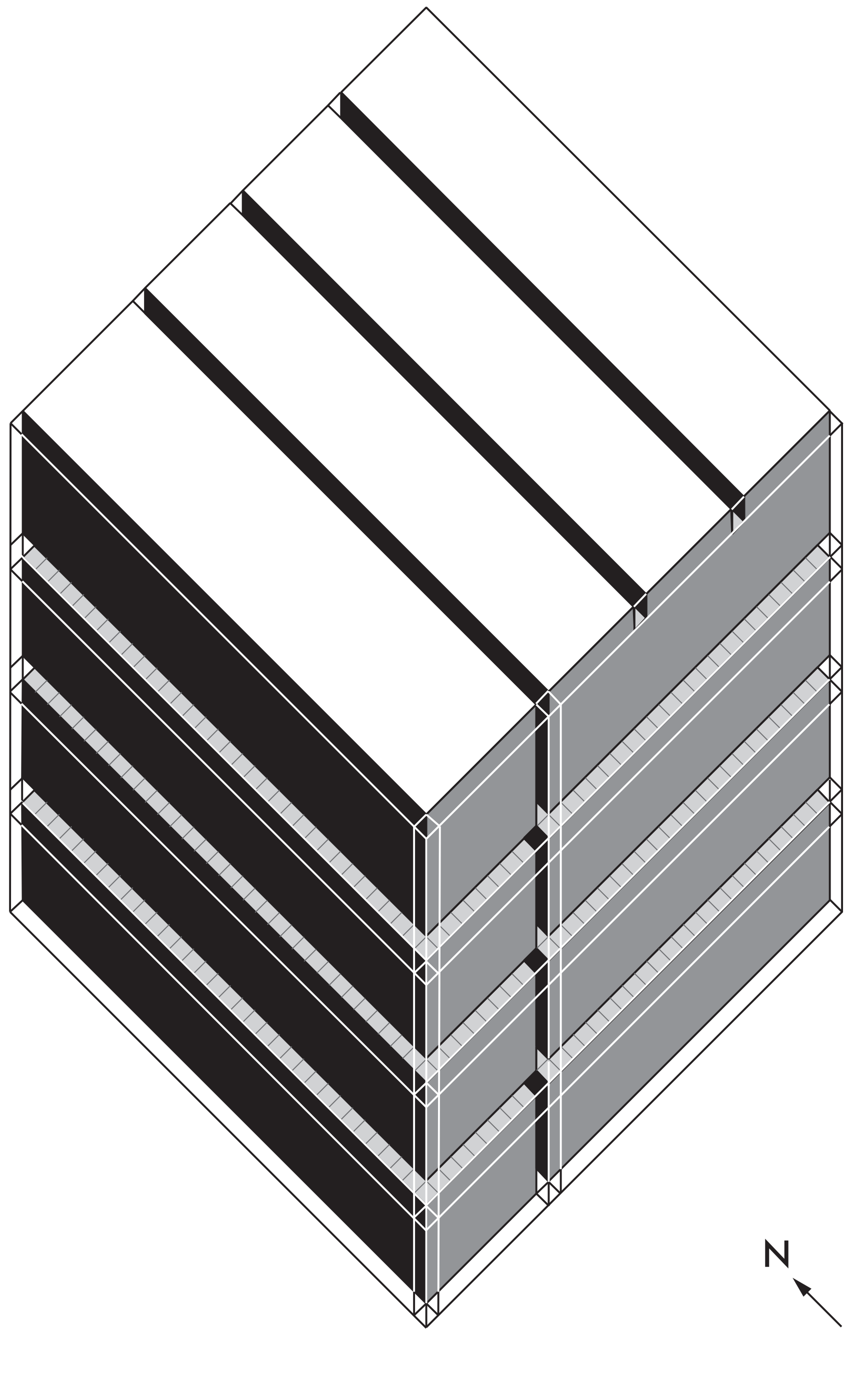
Some are more critical of the modern style for its excessive formality (see the comments following 6 Axiomatic and 33 Calculus), while others feel that it isn't formal enough (see 18 Indented).

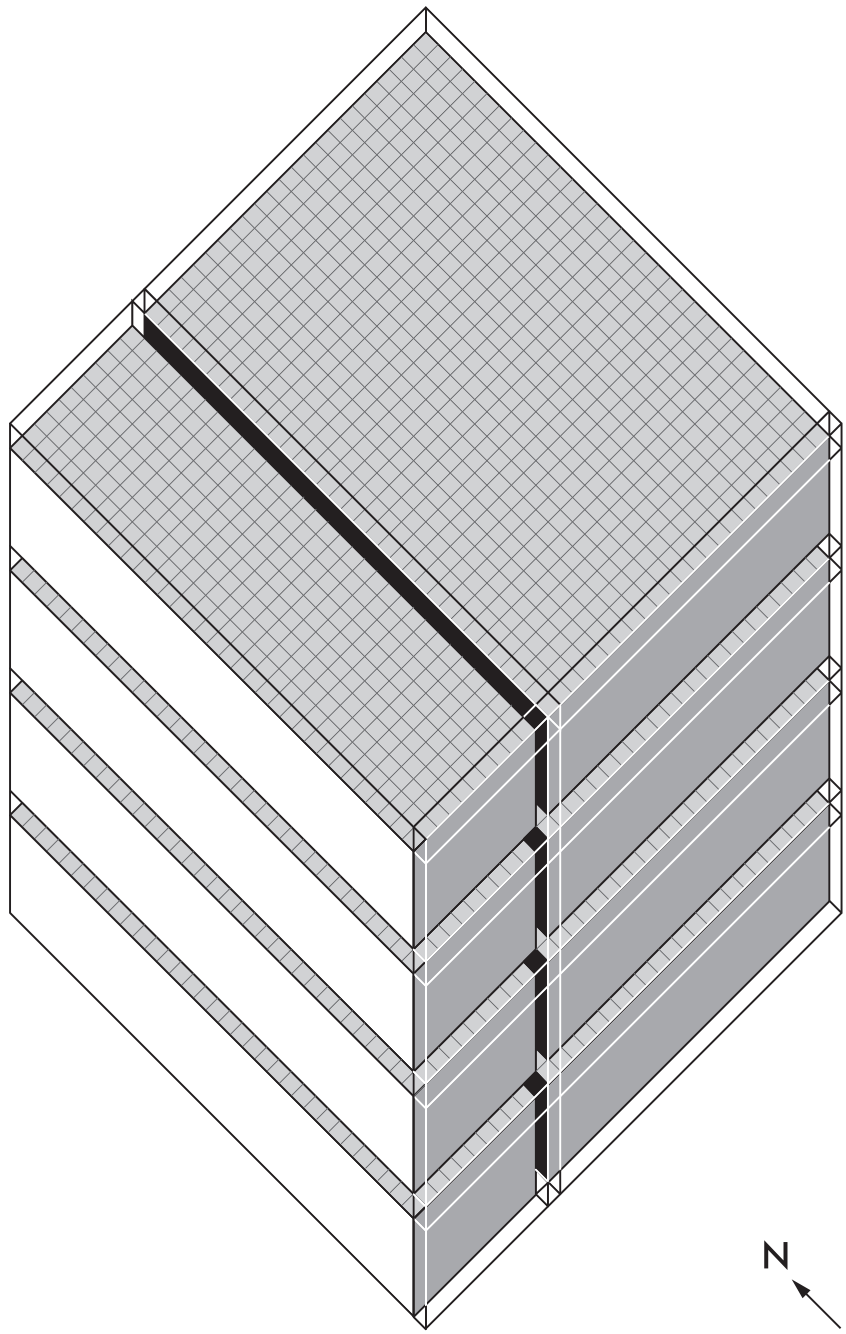
In this abstract conception of algebra, a polynomial equation (cubic or other) is regarded as but a number or a point in the infinite totality of polynomials. That totality, denoted $\mathbb{R}[X]$, is called the ring of polynomials in the variable X, and it shares many of the properties and operations enjoyed by more familiar number systems such as the integers. For example, there is a natural way to add and multiply pairs of polynomials. A principal ideal $I = (p)$ generated by a fixed polynomial p is the set of polynomials obtained by multiplying p by any and all polynomials in $\mathbb{R}[X]$.

Like the overtones of a plucked string, the principal ideal expresses the relationship between p and the ring $\mathbb{R}[X]$ containing it. One way to analyze p itself is to declare all elements of I to be equal to zero. One thus obtains the quotient, denoted $\mathbb{R}[X]/I$, which is also a ring, equipped with addition and multiplication operations and a zero element. Unlike the integers or the ring of polynomials, however, the ring $\mathbb{R}[X]/I$ may include nonzero elements whose product is zero. The modern solution of the cubic here relies on the fact that from this abstract point of view, one finds among these "zero divisors" the needed factors of p. This idea is the starting point of modern algebraic geometry, which remains a very active area of research.[116]

62

Axonometric




N

In his study of Italian architect Giuseppe Terragni, the high modernist American architect Peter Eisenman identifies two operative conceptualizations of space:

> The first considers space as subtractive, or cut away from a solid.... The second conception of space, which has Renaissance antecedents, considers space as additive, made up of a series of implied layers, much like a deck of cards.... The initial marking of a specific form can be considered as either additive, if one is filling up the void, or subtractive, if one is cutting away from the solid.[117]

These diagrams apply the subtractive conception of space to envision our proof as two different studies in massing, inspired by Terragni's 1936 Casa del Fascio in Como.

The first diagram depicts the volume $x^3-6x^2+11x-6-(2x-2)$ as the product $(x-1)\times(x-2)\times(x-3)$ less twice the product $(x-1)\times 1\times 1$, where the three factors in each term measure, respectively, the extension in the North-South direction, East-West direction, and elevation. The two black gutters incised in the roof represent the subtraction of twice $(x-1)\times 1\times 1$.

The second diagram depicts the same volume as the product $(x-1)\times(x-1)\times(x-4)$. To see the equivalence between the two diagrams, imagine removing the roof and attaching it to the West façade.

For another volumetric analysis, see 10 Wordless.

63

Back of the Envelope

Back of the Envelope

The Italian physicist, Enrico Fermi was particularly adept at making quick and surprisingly accurate estimates with no more detail than the blank side of an envelope or other scrap of paper could accommodate. So-called Fermi questions require students to practice similar estimates based on very few givens.[118] A typical example is "How many piano tuners are there in the city of Chicago?"

The brief solution here proceeds by guessing $x = 1$ and then dividing the cubic polynomial by the factor $x - 1$ using a shorthand version of long division for polynomials called *synthetic division*.

$x^3 - 6x^2 + 9x - 4$ roots 1, 4

$1 - 6 + 9 - 4 = 0$

$$\begin{array}{c|cccc} & 1 & -6 & 9 & -4 \\ & & 1 & -5 & 4 \\ \hline 1 & 1 & -5 & 4 & 0 \end{array}$$

$x^2 - 5x + 4$

$(x-1)(x-4)$

Theorem (L.). *The roots of* $P = x^3 - 6x^2 + 11x - 6 - 2x + 2$ *over* $\mathbb{R}$ *are* $x = 1, 4$.

Let $S^3(k^{2^*})$ be the vector space of binary cubics over k with $Char(k) \neq 2, 3$.

Def. Symplectic structure on $S^3(k^{2^*})$:
For $P = ax^3 + 3bx^2y + 3cxy^2 + dy^3$ and $P' = a'x^3 + 3b'x^2y + 3c'xy^2 + d'y^3$, closed non-degenerate 2-form

$$\omega(P, P') = ad' - da' - 3bc' + 3cb'.$$

Lie algebra $\mathfrak{sl}(2, k)$ acts on k^{2^*}. Moment map $\mu : S^3(k^{2^*}) \to \mathfrak{sl}(2, k)$:

$$\mu(P) = \begin{pmatrix} ad - bc & 2(bd - c^2) \\ 2(b^2 - ac) & -(ad - bc) \end{pmatrix}.$$

Normed square of moment map:

$$Q_n(P) = -\det \mu(P).$$

Prop. (Slupinski-Stanton, 2012). If $P = ax^3 + 3bx^2y + 3cxy^2 + dy^3 \neq 0$ over k, $ad - bc \neq 0$, and $Q_n(P) = 0$. Then

$$P = \left(-(b^2 - ac)x + \frac{1}{2}(ad - bc)y\right)^2 \left(\frac{a}{(b^2 - ac)^2}\,x + \frac{4d}{(ad - bc)^2}\,y\right).$$

Proof of Thm. Tschirnhaus trans. $x \mapsto \xi + 2$, $P \mapsto P' = \xi^3 - 3\xi^2 - 2$.
Homogenize $P'' = \xi^3 - 3\xi^2 y - 2y^3$. $\mu(P'') = \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix}$, $Q_n(P'') = 0$.

$$\begin{aligned} \text{Prop.} \Rightarrow P'' &= (\xi + y)^2(\xi - 2y) \\ \Rightarrow P' &= (\xi + 1)^2(\xi - 2) \\ \Rightarrow P &= (x - 1)^2(x - 4). \end{aligned}$$

Research Seminar

University mathematics departments typically hold weekly seminars in several different research areas such as number theory, analysis, topology and geometry, combinatorics, etc. The speakers, whether local or visiting from another math department, present their recent findings. Such a talk may begin with a few pleasantries or background remarks before turning sharply toward the extremely specialized formalism that is characteristic of modern mathematics.

Using a chalkboard, digital projector, or, less frequently, an overhead projector, the speaker outlines the technical details in a sequence of definitions, theorems, lemmas (intermediate theorems), and one or more proofs. The text, being supplemented by the speaker's words, tends to be somewhat abbreviated, as is the style here. Beyond the first five to ten minutes, many talks are impenetrable to nonspecialists. Occasionally an audience member will, regardless of specialty, doze off. After the talk there is applause, one or more questions, then another round of applause. The careers of all but the most reclusive of research mathematicians are sustained by attending and giving such talks.

Gautam Chinta of The City College of New York directed my attention to the paper entitled *The special symplectic structure of binary cubics*.[119] The authors, Marcus J. Slupinski of Université Louis Pasteur in Strasbourg and Robert J. Stanton of Ohio State University, derive the Cardano formula for a real root of an arbitrary cubic, of which the "theorem" presented here is a trivial application. Conventions of modesty deem that a speaker may take credit for a result by labeling it with the first initial of his or her last name. Thus a theorem by, for example, François Le Lionnais would appear as shown, or maybe *Le L.*, I'm not sure.

Talks are sometimes followed or preceded by a more informal discussion over lunch, dinner, or 65 Tea.

65
Tea

ALPHA: That was a very nice talk.

LAMBDA: Thank you.

BETA: I enjoyed the talk too even though I confess I know nothing about symplectic geometry. I was curious to hear about its application to a cubic like this *x* cubed minus…what was it?

LAMBDA: You can think of it as just *x* cubed minus six *x* squared plus nine *x* minus four.

ALPHA: The symplectic structure is inherited functorially from the vector space *k* squared, isn't it?

LAMBDA: That's right.

DELTA: Hi Professor Zeta!

ZETA: Uh-huh.

GAMMA: I'm very sorry that I wasn't able to attend the talk—I had to go to a defense in the physics department. Speaking of physics, did you get into classical mechanics?

BETA: Huh? I guess I really wasn't paying attention.

EPSILON (*to* DELTA): Here's one for you: What's purple and commutes?

DELTA (*to* EPSILON): Oh, please.

ALPHA: If you model the motion of an object in terms of its position and momentum, these variables range smoothly over a manifold endowed with a symplectic form—basically a means to integrate area. Only area isn't just area, it's actually a conserved quantity that expresses one or another of the physical laws….

ZETA: Who ate all the cookies?!

ALPHA: The secretary is just about to bring them out, Professor.

ZETA: Oh. Well, I didn't think it was you. Some of these grad students….

GAMMA: A Hamiltonian actually came up in the dissertation defense this morning.

LAMBDA: I didn't get into the physics, but Slupinsky and Stanton's work that I used is very much motivated by questions about the symplectic geometry of Heisenberg graded Lie algebras.

GAMMA: Aha, quantum mechanics!

BETA: So, what I did follow was that you were able to factor the cubic once you knew the value of these symplectic objects, the moment map, and its norm square. Why is that?

ALPHA: Isn't the norm square essentially the discriminant?

LAMBDA: That's right. The norm square is negative the discriminant divided by twenty-seven.

BETA: I see. So, if the norm square of the moment map vanishes, the discriminant does too, and you get a repeated root. I learned something. Thank you!

LAMBDA: You're very welcome.

ZETA: An abelian grape!

Tea

Many university mathematics departments hold tea in the afternoon where a number of faculty, their post-docs, graduate students, and occasionally undergraduates will appear. Sometimes a research seminar will be scheduled near tea time, and the common room or lounge where tea is served offers a venue for informal discussion. William Thurston contrasts the formal and informal modes of communication: "In [research] talks, people are more inhibited and more formal. Mathematical audiences are often not very good at asking the questions that are on most people's minds, and speakers often have an unrealistic preset outline that inhibits them from addressing questions even when they are asked." In comparison, Thurston notes that "One-on-one, people use wide channels of communication that go far beyond formal mathematical language. They use gestures, they draw pictures and diagrams, they make sound effects and use body language."[120]

In case you want to see how this actually happens, at least one ethnographer found answers by taking up residence in the tea room.[121] This variation imagines how such a conversation might go following LAMBDA's 64 Research Seminar talk. The dynamics of rank, seniority, and other hierarchies are more pronounced in some departments than others. I don't know the origin of the pun EPSILON sets up for ZETA, but puns and other lousy math jokes are one way you know that you're in a mathematics department.

66

Hand Waving

The expression “proof by handwaving” is usually used in an ironic or derogatory sense. Rota describes an instance of hand waving in his recollection of a particular professor: “His lectures were loud and entertaining. He wrote very large on the blackboard, in a beautiful Italianate handwriting with lots of whirls. Sometimes only one huge formula appeared on the blackboard during the entire period; the rest was handwaving. His proofs—insofar as one can speak of proofs—were often deficient. Nonetheless, they were convincing, and the results became unforgettably clear after he had explained them. The main idea was never wrong.”[122]

The point of departure for cognitive scientists Rafael Núñez and George Lakoff in their somewhat controversial analysis of mathematical ideas is the principle that such ideas are based in sensory-motor experience. One example of an embodied cognitive mechanism that they present is the Source-Path-Goal schema, and they argue that this schema governs mathematical thinking “when we think of two lines ‘*meeting* at a point’... or the graph of a function as ‘*reaching* a minimum at zero.’ ”[123] For Lakoff and Núñez, to understand the meaning of mathematics is to deconstruct conceptual metaphors like these.

This variation imagines a gestural expression of the cubic equation as the nineteenth century French scientist Étienne-Jules Marey might have captured it in one of his chronophotographs.

67

Approximate

Suppose $x^3-6x^2+11x-6=2x-2$ for an unknown real number x. Notice that the left-hand side is the familiar product $(x-1)(x-2)(x-3)$, and after subtracting three times the right-hand side from the left-hand side we have another simple factorization:

$$x^3-6x^2+11x-6-3(2x-2)=x^3-6x^2+5x=x(x-1)(x-5).$$

In the one-parameter family of polynomial equations $x^3-6x^2+11x-6=t(2x-2)$, the original equation lies at $t=1$, between the equations at $t=0$ and $t=3$ that were just factored. The ordered set of solutions of the given equation is thus approximated by the neighboring solutions, $(a_0,b_0,c_0)=(1,2,3)$ and $(a_3,b_3,c_3)=(1,0,5)$, as $(a_1,b_1,c_1)=(1,1,4)$. In fact, these approximations do solve the equation, so $x=1$ or $x=4$.

The argument here is "approximate" in two ways: the method employed is one of approximation—the solution is interpolated from the solutions of two related cubic equations—and the missing details of the proof make the argument nearly rigorous, but not exactly so. Of course, this is not to imply that approximation theory is any less rigorous than another area of mathematics. Typically, mathematicians will refer to such a proof as an "informal proof" or "proof sketch," and this collection includes further examples of such proofs besides this one.

Among the questions left open by the proof are: How are the solutions in an ordered set of solutions ordered? Given that the parameter $t=1$ is closer to $t=0$ than it is to $t=3$, wouldn't a better approximation to the solution be nearer to $(1,2,3)$ than $(1,0,5)$? Is it reasonable to interpolate a solution given the possibility that even a very small change in the coefficients of a polynomial could lead to a significant change of its roots?

The problem raised by this last question is expressed by saying that for polynomials, root approximation from coefficients is an *ill-conditioned* problem. The sensitivity of approximating a *double* root from its coefficients is expected (see 84 Tabular), but polynomials with well-separated roots are not immune either. The British mathematician James Wilkinson, who discovered the first such example (Wilkinson's polynomial), explained why this was so shocking:

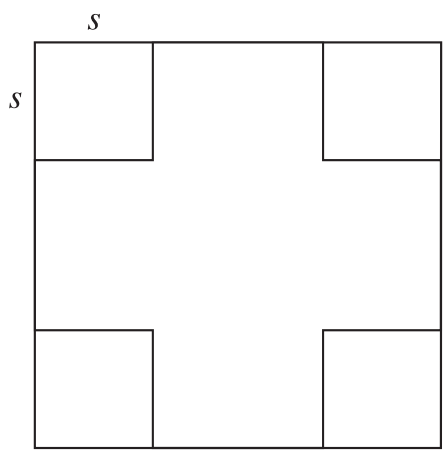
> The history of mathematics has, in a sense, contrived to make mathematicians feel "at home" with polynomials... polynomials are such agreeable functions. The cozy relationship that mathematicians enjoyed with polynomials suffered a severe setback in the early fifties when electronic computers came into general use. Speaking for myself I regard it as the most traumatic experience in my career as a numerical analyst.[124]

A small box is to be made by cutting out four equal squares from a sheet of cardboard that measures 6 in by 6 in and then folding the resulting flaps. If the box is to have a volume of 16 in^3, show that the length of the sides of the cut-away squares must be 1 in.

68

Word Problem

Solution. Let s be the length measured in inches of the side of the cut-away square, as shown in the diagram. Our task is to show that $s = 1$. After cutting away a square from each corner, the width of each flap is $6 - 2s$, since the sides of the cardboard sheet were originally 6 in before we removed a square from each of the two corners joined by a side. The depth of the box equals the depth of the flap, which is s.



Now we can write down an equation for the volume of the box in terms of the dimensions of the flap:

$$V = \text{length} \times \text{width} \times \text{depth} = (6 - 2s)(6 - 2s)s.$$

The problem stipulates that the volume V must equal 16, therefore, our task is to solve the equation

$$(6 - 2s)(6 - 2s)s = 16.$$

This equation is equivalent to the degree three equation

$$s^3 - 6s^2 + 9s - 4 = 0,$$

which has two distinct roots, $s = 1$ and $s = 4$. Notice however that a valid solution must satisfy the inequality $s < 3$, otherwise the squares would cover the perimeter of the sheet and there would be no flaps to fold. Thus $s = 1$ is the only valid solution to the problem.

Word Problem

Textbook publishers notwithstanding, everyone knows that word problems are contrived. How you feel about the contrivance probably reflects your current teaching load as much as your mathematical taste. Some find them to be “a valuable form of mathematical modelling.”[125] Others view them as an abomination: “The real lesson being imparted is that mathematics is a stupid, arbitrary subject having no relevance to the real world.”[126] For a more dispassionate perspective, see the note after 5 Puzzle.

What surprises me about word problems is that they have a long and illustrious lineage that extends across cultures and through the entire history of mathematics (including Cardano; one of the many word problems in *Ars Magna* was a source for the challenge in 43 Screenplay). Maybe textbooks would be less dry if they sampled more broadly from this history. Consider this one from the Līlāvatī by the twelfth century Indian mathematician Bhāskara II:

> Whilst making love a necklace broke.
> A row of pearls mislaid.
> One sixth fell to the floor.
> One fifth upon the bed.
> The young woman saved one third of them.
> One tenth were caught by her lover.
> If six pearls remained upon the string
> How many pearls were there altogether?[127]

69

Statistical

Suppose that $x^3-6x^2+11x-6=2x-2$ has three real solutions x_1, x_2, x_3, with mean $\overline{x}$ and standard deviation s

$$\overline{x}=\frac{x_1+x_2+x_3}{3} \qquad s=\sqrt{\frac{(x_1-\overline{x})^2+(x_2-\overline{x})^2+(x_3-\overline{x})^2}{3}}. \tag{1}$$

We can bound the roots by considering how many standard deviations any sample x can be from the mean $\overline{x}$. After some thought it's clear that a single root will deviate the most from the mean when both of the other roots coincide. Suppose $x_2=x_3$ and x_1 is the outlier. The equations for the mean and standard deviation (1) imply that $x_2=\frac{3}{2}\overline{x}-\frac{1}{2}x_1$, and hence

$$s^2=\frac{(x_1-\overline{x})^2+2\left(\left(\frac{3}{2}\overline{x}-\frac{1}{2}x_1\right)-\overline{x}\right)^2}{3}=\frac{(x_1-\overline{x})^2}{2}.$$

Thus $x_1=\overline{x}\pm\sqrt{2}s$, and, for any sample x,

$$\overline{x}-\sqrt{2}s\leq x\leq\overline{x}+\sqrt{2}s. \tag{2}$$

To compute the values of the mean and standard deviation, we apply the Vietà formulas that express the linear and quadratic coefficients of a polynomial in terms of its roots:

$$x_1x_2+x_2x_3+x_1x_3=a_1,$$
$$x_1+x_2+x_3=-a_2.$$

Our cubic takes the standard form $x^3-6x^2+9x-4=0$, with $a_1=9$ and $a_2=-6$. Hence, the mean is $\overline{x}=-a_2/3=2$, and, by expanding the expression for s in equation (1), we have

$$\begin{aligned} s&=\sqrt{\frac{x_1^2+x_2^2+x_3^2-2\overline{x}(x_1+x_2+x_3)+3\overline{x}^2}{3}} \\ &=\sqrt{\frac{(x_1+x_2+x_3)^2-2(x_1x_2+x_2x_3+x_1x_3)-2\overline{x}(x_1+x_2+x_3)+3\overline{x}^2}{3}} \\ &=\sqrt{\frac{2a_2^2-6a_1}{9}} = \sqrt{2}. \end{aligned}$$

Finally, by substituting $\overline{x}=2$ and $s=\sqrt{2}$ into the inequality (2), we obtain bounds on the supposed three real roots

$$0\leq x\leq 4.$$

By comparison with the previous variations, this proof achieves a very weak result—it begins with an unproven assumption that all roots are real (they need not be) and it ends with only an *estimate* of a solution. But I find it a striking example of the way tools developed to study the "messy" world outside mathematics can be used to examine the idealized objects of "pure" mathematics.

This proof, like 78 Probabilistic, demonstrates the Laguerre-Samuelson inequality $|x| \leq \overline{x} \pm s\sqrt{n-1}$ in the case of a polynomial $n=3$. The statistical interpretation of the inequality is due to economist Paul Samuelson, who answered the titular question "How Deviant Can You Be?" of his 1968 article by showing that the distance between the mean and any individual in a population of N can be at most $\sqrt{N-1}$ standard deviations.[128] The argument is patterned on the informal proof that Samuelson presents in the introduction of his paper; in particular, I do not prove the key observation ("After some thought...") as Samuelson does in the body of his paper. Neither have I attempted to reproduce his *style* of writing, which strikes me as more lively than is typical in statistics, or, for that matter, mathematics on the whole.

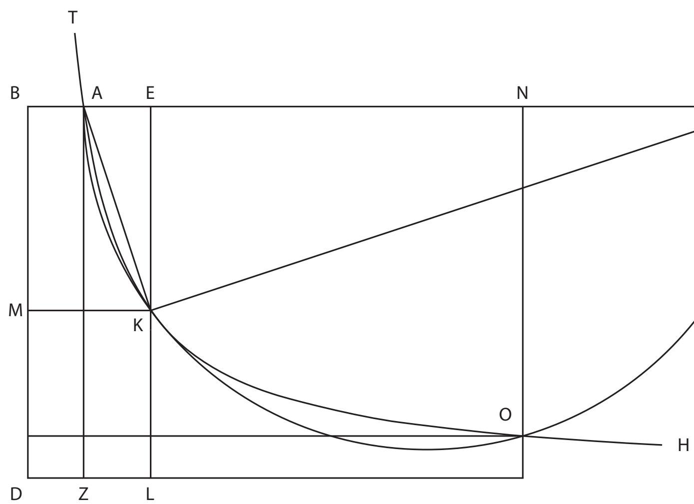
In the name of God, gracious and merciful! I shall demonstrate a geometrical solution to the tetranomial equation in which a cube and 9 sides are equal to 6 squares and 4. The proof of this equation is only possible by reference to the properties of conic sections, and it should be understood that this demonstration cannot be comprehended except by those who know thoroughly Euclid's books as well as Apollonius' work on conics.

70

Another Medieval

Let *BC* be equal to 6, the given number of squares and *BD* equal to the side of a square equal in area to 9, the number of sides and let *BD* be perpendicular to *BC*. Construct a solid equal in volume to 4, the given number and having as base the square of *BD* and height equal to the segment *BA* taken on *BC*.

Complete the rectangle *ABDZ*, and describe on *AC* as a diameter a circle *AKC* whose position will be known. Pass through the point *A* a hyperbola with *BD* and *DZ* as asymptotes. It will be the conic *TAH*, which is of known position.



Since the tangent to a hyperbola cuts both of its asymptotes, the tangent to *TAH* at *A* cannot coincide with *AZ* or the tangent to *AKC* at *A*, which are parallel to asymptote *BD*. Therefore, the hyperbola enters the circle at *A* and necessarily meets the circle in at least one other point. Let *K* be a second point of intersection.

Drop from this point two perpendiculars *KM* and *KE* on *BC* and *BD*. The segment *BE* will be the side of the required cube, as we will now prove.

Complete the rectangle *KD* and the right triangle *AKC*. The rectangle *AD* will be equal to the rectangle *KD*, as expounded by Apollonius in the twelfth proposition of the second book. Subtract from *AD* and *KD* the common rectangle *MZ* and add to both the rectangle *AK*. Then *BK* will be equal to *AL*. The sides of these two rectangles, and in the same manner the squares of their sides, will be reciprocally proportional.

But KE is to EA as EC is to KE since triangle AEK and triangle KEC are similar. Consequently, the square of KE is to the square of EA as EC is to EA. Given the reciprocal proportion above and that BD equals LE, it follows that the solid whose base is the square of BD and whose height is EA is equal to the solid whose base is the square of BE and whose height is EC.

Let the cube of BE and the solid whose base is the square BD and whose height is BA be added to both. Then the cube of BE plus the solid whose base is the square of BD (equal to 9) and whose height is BE will be equal to the solid whose base is the square of BE and whose height is BC (equal to 6) plus the solid whose base is the square BD and whose height is BA (equal to 4), which is what was required.

The same proof shows that for the third point of intersection O between AKC and TAH, the segment BN cut by the perpendicular ON is another geometric solution. It is now time that we should conclude this demonstration with gratitude to God and praising all of His prophets.

This historical style could have been named “medieval Islamic,” a designation that the historian of mathematics Len Berggren applies to mathematical works originating in the Islamic civilization that dates from 750 to 1450.[129] This culture created algebra and made major advances in trigonometry, numerical analysis, and astronomy, among other areas.

The geometric construction here is due to the Persian poet-mathematician, Omar Khayyam, who lived during the eleventh and twelfth centuries.[130] See 71 Blog for a discussion of this proof in modern algebraic notation. Here are a couple notes on the details of the construction: regarding the reference to Apollonius, Heath’s translation of the *Conics* II.12 reads as follows:[131]

> If Q, q be any two points on a hyperbola, and parallel straight lines QH, qh be drawn to meet one asymptote at any angle, and QK, qk (also parallel to one another) meet the other asymptote at any angle, then
>
> $$HQ \cdot QK = hq \cdot qk.$$

The fact that triangle AEK and triangle KEC are similar can be deduced from the fact that AKC is a right triangle.

While his *Treatise on Demonstration of Problems of Algebra* presents geometric solutions for every type of cubic equation, Khayyam was aware of the importance of numerical solutions, writing: “If the object of the problem is an absolute number, neither we, nor any of the algebraists have succeeded, except in the case of the first three degrees, namely number, thing and square, but maybe those after us will.”[132]

How do you say “algebra” in ancient Greek?

May 18, 2012

71

Blog

At the end of my last post about the ingenious 16th century solution of the cubic equation

$$x^3+9x=6x^2+4$$

I posed the question: why didn’t anyone at that time try solving this equation as the intersection of two algebraic curves? My guess was that such an approach called for more sophisticated algebraic notation, if not Cartesian coordinates, none of which were available at the time. Recently a reader posted the answer, and it wasn’t what I was expecting. Why isn’t there a Renaissance solution of the cubic by intersecting curves? *Because that approach was mastered four centuries earlier!*

It was Omar Khayyam, the poet and mathematician who lived from 1048 to 1131 in Nishapur (present day Iran), who constructed the solutions of every species of cubic equation using pairs of conics sections.

Wanting to find out more about this method, I got out Dieudonné’s *History of Algebraic Geometry* that my advisor gifted me after passing my orals. “That’s funny,” I think. Nothing on Khayyam or his work in this entire book. For a moment I feel relieved of my embarrassment for overlooking Khayyam, like “well, at least I’m in good company.”

But reading more closely, I start to feel less good about that company. In a section entitled “The First Epoch: Prehistory (ca. 400 B.C.–1630 A.D.)” I find the curious statement: “Apollonius’ theorems translate immediately in our notation into the equation of the evolute that only the underdeveloped state of Greek algebra prevents him from writing.”

Which brings me to the title of this post: What’s the ancient Greek word for “algebra”?

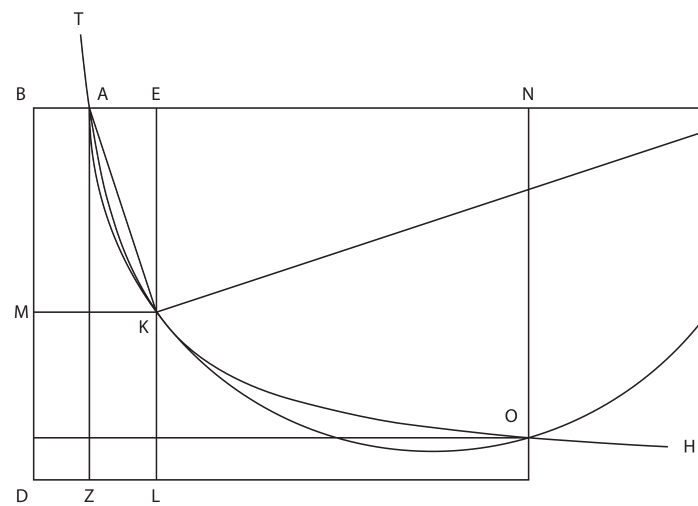
I’ll give you a hint. . . they didn’t have one! “Algebra” as I’m sure most of you know, comes from the Arabic word “al-jabr,” which is part of the title of the first book on the subject by the 9th century mathematician al-Khwarizmi.

Even if we allow that Dieudonné was preoccupied with the *modern* notion of algebraic geometry, it doesn’t excuse the book for lending support to the myth that math, including algebra no less, is a product of the Western world. By jumping from Apollonius to Descartes without citing the contribution of a single

Arab, the book tacitly affirms the Orientalist view that “. . . Arabic science only reproduced the teachings received from Greek science.”

People, we call it *algebra* for a reason!

Now. Ahem. Without further ado, let me show how Khayyam would have solved this cubic, according to his book of Algebra. The key idea is to intersect hyperbola *TAH* with circle *AKC*, as shown below. The solutions 1, 4 correspond to the lengths of the line segments *BE* and *BN*.



Here’s the set up: The coefficients of the cubic are encoded as $6=BC$ for the quadratic term, $9=(BD)^2$ for the linear term, and $4=(BD)^2\cdot BA$ for the constant. And the conics are constructed so that both pass through *A*, the circle’s diameter is *AC*, and the hyperbola’s asymptotes are *BD* and *DZ*. Dropping perpendiculars from each of the other intersection points *K* and *O* to *AC* produces the solutions. How on Earth, you ask? Let’s show that

$$(BE)^3+9BE=6(BE)^2+4.$$

Not surprisingly, we’ll put to use some properties of those conics. The hyperbola gives us the relation $KE\cdot BE=LE\cdot EA$, which implies $(KE)^2/(EA)^2=(LE)^2/(BE)^2$, and the circle gives us $KE/EA=EC/KE$, which implies $(KE)^2/(EA)^2=EC/EA$. Given that $LE=BD$, the above combines to produce

$$(BD)^2\cdot EA=(BE)^2\cdot EC.$$

Now add $(BE)^3+(BD)^2\cdot BA$ to both sides and factor:

$$(BE)^3+(BD)^2\cdot BA+(BD)^2\cdot EA=(BE)^3+(BD)^2\cdot BA+(BE)^2\cdot EC$$

$$(BE)^3+(BD)^2\cdot(BA+EA)=(BE+EC)\cdot(BE)^2+(BD)^2\cdot BA$$
$$(BE)^3+(BD)^2\cdot BE=BC\cdot(BE)^2+(BD)^2\cdot BA.$$

But this is exactly our cubic equation in terms of *BE* once you restore the original coefficients. The same reasoning shows that *BN* is a root, too.

How could such beautiful work ever be neglected?

Just because some mathematicians like to think of themselves as being above the fray doesn't mean that our history isn't "written by the victors."

Blog

Blogs (or weblogs) are a fantastic way to put off writing, grading, or sleep. Unlike the hours eaten up by other forms of procrastination, the time spent on blogs might even leave you feeling smarter—especially if the blogger is an expert in their field.

In all seriousness, the case could be made that blogs represent a new genre of mathematical literature. Informal expository writing has long been available offline in one of a very small number of monthly or quarterly math magazines, but those outlets rarely feel as current, opinionated, or free-form as blogs do. And, as in every sector of media, nothing in print approaches the level of engagement made possible by an active comments section. Blogs might even deserve credit for a new approach to mathematics *research*—see 25 Open Collaborative.

What blogs also offer the math community is a much needed means of self reflection and critique. Witness the American Mathematical Society's *inclusion/exclusion* blog, Izabella Laba's *The Accidental Mathematician*, or *Dr. Z's Opinions* by Doron Zeilberger. Reflection and critique are what readers expect from blogs, but the mathematics community, with its apprentice-like professional training and deep faith in meritocracy, has been at times slow to surface criticism.

For the same geometric construction in this blog post but in the medieval Islamic style, see 70 Another Medieval. I learned about Dieudonné's omission[133] from science historian Roshdi Rashed's "The Notion of Western Science: 'Science as a Western Phenomenon.' "[134] The quote beginning "…Arabic science," due to nineteenth century French physicist and science historian Pierre Duhem, is also quoted by Rashed.[135]

Théorème. *Les racines réelles de* $P(x)=(x^3-6x^2+11x-6)-(2x-2)$ *sont 1 et 4.*

Démonstration. On vérifie immédiatement que $P(1)=P(4)=0$. Comme le polynôme $P\in\mathbb{R}[x]\subset\mathbb{C}[x]$ est de degré 3, d'après le théorème de d'Alembert on sait qu'il admet au maximum 3 racines réelles. Nous raisonnons par l'absurde en supposant que $a\in\mathbb{R}$ soit une troisième racine distincte. Alors,

$$P(x)=(x-1)(x-4)(x-a)=x^3-(5+a)x^2+(4+5a)x-4a.$$

On déduit de la seconde égalité que $a=1$, ce qui nous donne une contradiction. Donc, les racines de P sont 1 et 4. CQFD

72

Translated

Despite the conception of mathematics as a "universal language," mathematicians report their discoveries using natural language, and that language is overwhelmingly English. While publishers will state pragmatically that writing in English ensures the broadest readership, it may not be the language of choice. Only a few of the international journals topping the citation rankings state in their instructions to authors that they accept papers written in a language other than English, the alternatives usually being French and German. With perhaps one exception (the *Publications Mathématiques* of the *Institut des Hautes Études Scientifiques*), non-English papers rarely appear in these journals. This was not always the case. A graduate student in mathematics anywhere will encounter references to non-English articles, and many PhD programs in America require reading knowledge of a foreign language. (Although this may change; while I was a graduate student, the foreign language requirement in my program went from reading knowledge of two languages, among French, German, and Russian, to just one.)

This variation and 73 Another Translated were rendered according to a rather naive constraint: first, the method of proof must rely on a theorem attributed to a mathematician identified with the culture in which the language dominates, and, second, the translation should avoid what the French call *les barbarismes* (i.e., words of foreign origin). The latter rule follows the wonderful style guide for mathematical writers *Conseils aux auteurs de textes mathématiques* by mathematician (and Oulipian) Michèle Audin.[136] If English continues to dominate mathematics publishing, mathematicians ought to incorporate more foreign words into their writing. At a minimum, let's update those stuffy Latinisms. *Raisonnement par l'absurde* sounds so much better than *reductio ad absurdum*, doesn't it?

Here is a literal translation of the French:

Theorem. The real roots of $P(x)=(x^3-6x^2+11x-6)-(2x-2)$ are 1 and 4.

Proof. One immediately checks that $P(1)=P(4)=0$. Since the polynomial $P\in\mathbb{R}[x]\subset\mathbb{C}[x]$ is of degree 3, after d'Alembert's theorem one knows that it admits at most 3 real roots. We proceed by *reductio ad absurdum* and suppose that $a\in\mathbb{R}$ is a distinct third root. Then,

$$P(x)=(x-1)(x-4)(x-a)=x^3-(5+a)x^2+(4+5a)x-4a.$$

One deduces from the second equality that $a=1$, which gives us a contradiction. So, the roots of P are 1 and 4. QED

Satz. Sei x ein ganze Zahl. Wenn $x^3-6x^2+11x-6=2x-2$ ist, dann ist $x=1$ oder $x=4$.

Beweis. Sei p das Polynom dritten Grades mit $p(x)=(x^3-6x^2+11x-6)-(2x-2)\in\mathbb{Z}[x]$. Wir wenden die Kroneckersche Methode an, um ein Polynom aus $\mathfrak{S}[x]$ in Primfaktoren zu zerlegen, wo $\mathfrak{S}$ ein Gauß'scher Ring ist. Soll nun $p(x)$ durch die Linearform $q(x)=x-a$ teilbar sein, so muss $p(x_0)$ durch $q(x_0)$ und $p(x_1)$ durch $q(x_1)$ teilbar sein. Jedes $p(x_i)$ in $\mathbb{Z}$ besitzt aber nur endlichviele Teiler. Setzen wir $x_0=2$, so ist $q(x_0)=2-a$ ein Faktor von $p(2)=-2$. Da ±1 und ±2 die einzigen Faktoren von -2 sind folgt, dass a ein Element der Menge $\mathfrak{M}_0=\{0,1,3,4\}$. Allerdings wenn $x_1=3$ ist, dann ist $3-a$ eines der Faktoren ±1, ±2, oder ±4 von $p(3)=-4$, m.a.W. es a in $\mathfrak{M}_1=\{-1,1,2,4,6,7\}$. Da eine Wurzel a von $p(x)$ beide Bedingungen erfüllen muß, schließen wir, wie erforderlich, dass $a\in\mathfrak{M}_0\cap\mathfrak{M}_1=\{1,4\}$.

Here is a literal translation of the German:

Theorem. Let x be an integer. If $x^3-6x^2+11x-6=2x-2$, then $x=1$ or $x=4$.

Proof. Let $p(x)$ be the polynomial of third degree $(x^3-6x^2+11x-6)-(2x-2)\in \mathbb{Z}[x]$. We apply Kronecker's method for decomposing a polynomial in $S[x]$ into prime factors, where S is a unique factorization domain. If $p(x)$ is divisible by the linear polynomial $q(x)=x-a$, then $p(x_0)$ must also be divisible by $q(x_0)$ and $p(x_1)$ by $q(x_1)$. But every $p(x_i)$ in $\mathbb{Z}$ has finitely many factors. If $x_0=2$, then $q(x_0)=2-a$ is a factor of $p(2)=-2$. Since ±1 and ±2 are the only factors of -2, it follows that a is an element of the set $M_0=\{0,1,3,4\}$. However, if $x_1=3$, then $3-a$ is one of the factors ±1, ±2, or ±4 of $p(3)=-4$; in other words, a is in $M_1=\{-1,1,2,4,6,7\}$. Since a root a of $p(x)$ must satisfy both conditions, we conclude $a\in M_0\cap M_1=\{1,4\}$, as required.

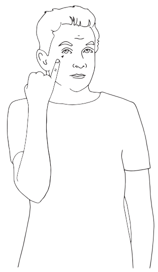
The proof is based closely on §25 Die Durchführung der Faktorzerlegung in endlichvieln Schritten of van der Waerden's *Moderne Algebra*, already cited in 61 Modern.[137]

In the sciences as a whole, linguists have noted the increasing dominance of "English as the international language of science." This trend is very likely reinforced by publication and citation practices. In an article entitled "The role of English in scientific communication: *lingua franca* or *Tyrannosaurus rex*?" Christine Tardy surveys this aspect of scientific culture from the perspectives of linguists and scientists. Citing one report, Tardy, who is professor of English at the University of Arizona, writes: "a coordinator of a leading Brazilian science journal project... stated that 'to publish in Portuguese would be a kind of provincialism.' "[138] It is not difficult to imagine both positive and negative impacts of English as the international language of mathematics on the discipline and its instruction.

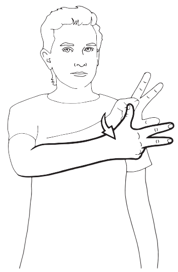
In the note to 72 Translated, I called the constraint that forms these styles "naive," and the reason is that it willfully ignores the question of *national* styles. That is a question worthy of engaging, to expose cultural and political influences in the development of mathematics. (An extreme and racialized example is the Nazi propaganda journal of mathematics *Deutsche Mathematik* [1936–1942], whose co-founder, Ludwig Bieberbach, wrote, "In my considerations I have tried to show that in mathematical activity there are issues of style and that therefore blood and race are influential in the way of mathematical creation."[139])

74

Another Interpreted

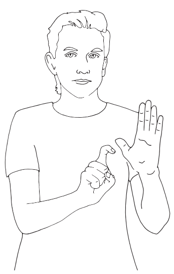

IF


X

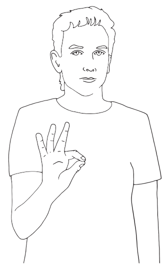

CUBED,

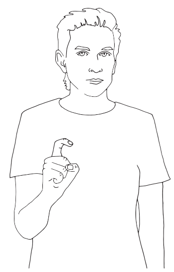

NEGATIVE

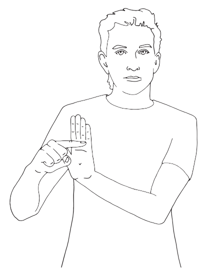

SIX


X

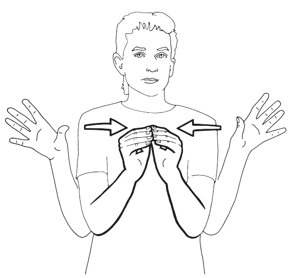

SQUARED,

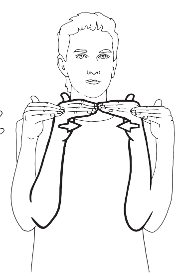

NINE


X,

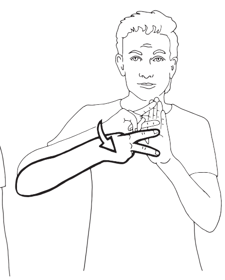

NEGATIVE

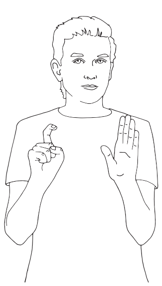

FOUR,

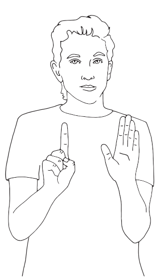

SUMMED

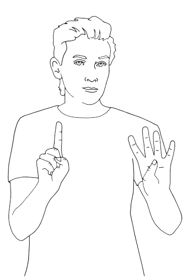

EQUALS

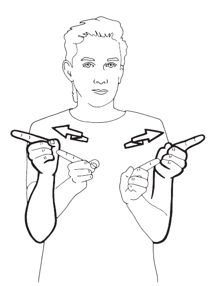

ZERO

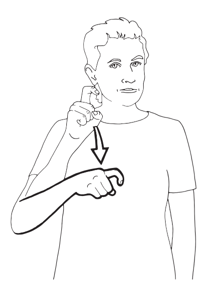

MEAN

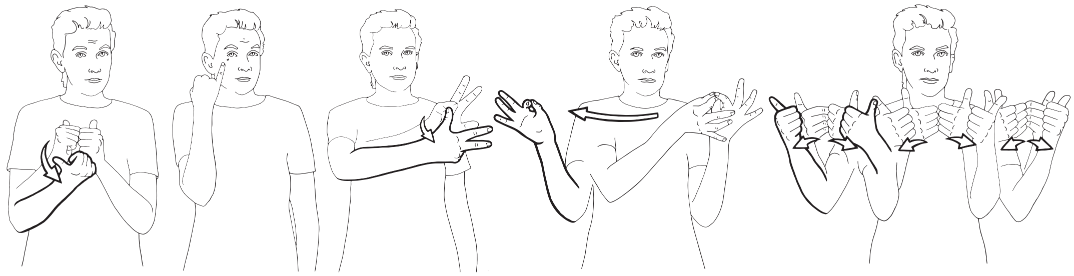

X


ONE

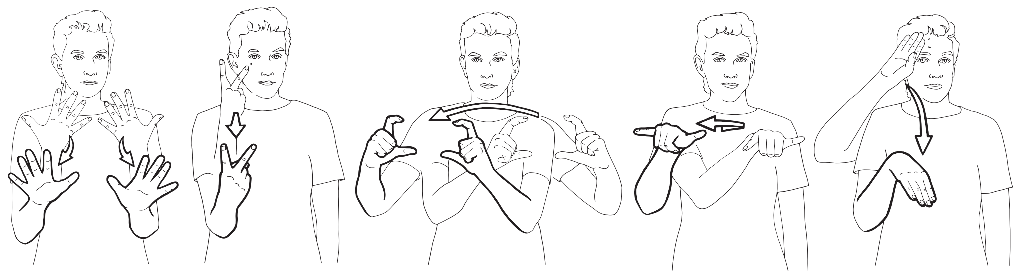

FOUR


ONE-OF-THOSE


MUST.



HOW? IF CUBED EXPRESSION FACTOR++



FINISH SEE (~) (~) SAME KNOW-THAT

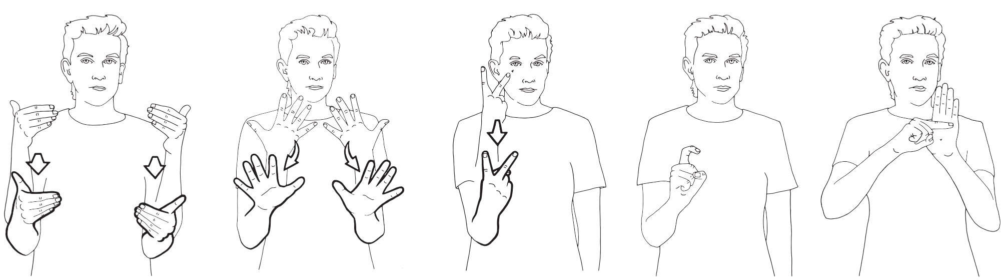
CUBED EXPRESSION REF1 DIFFERENTIATE FINISH

REF2 SQUARED EXPRESSION REF2 AUTOMATIC

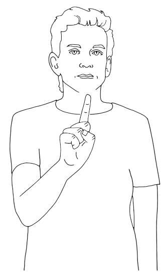
SEE (~) SAME CURIOUS GO-AHEAD

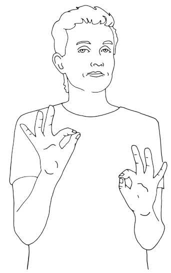
CHECK OUR CUBIC EXPRESSION DIFFERENTIATE,

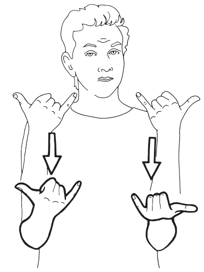
LEFT SQUARED EXPRESSION EASY FACTOR



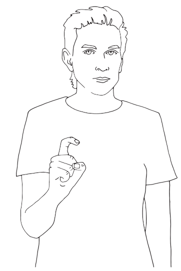
GO-AHEAD, FINISH SEE X MINUS


ONE,

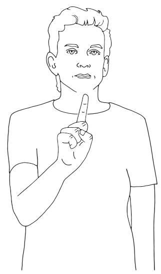

PERFECT.


NOW

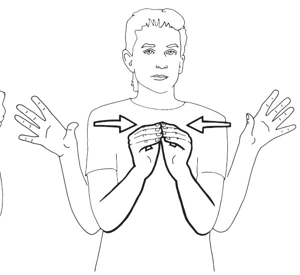

REF1,


X

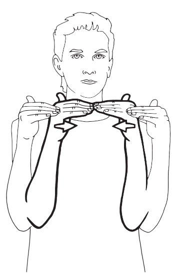

SUBSTITUTE


ONE

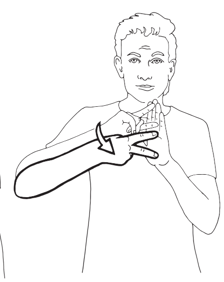

SUBSTITUTE++

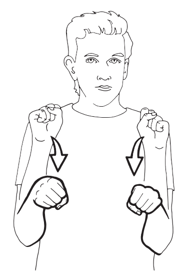

SUM


TRUEBUSINESS

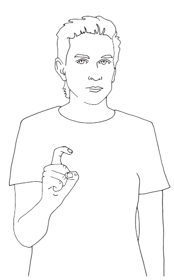

EQUAL

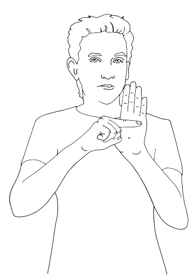

ZERO

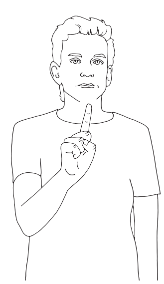

MEAN


CAN

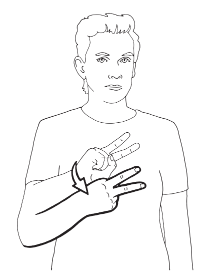

PULL-OFF


X

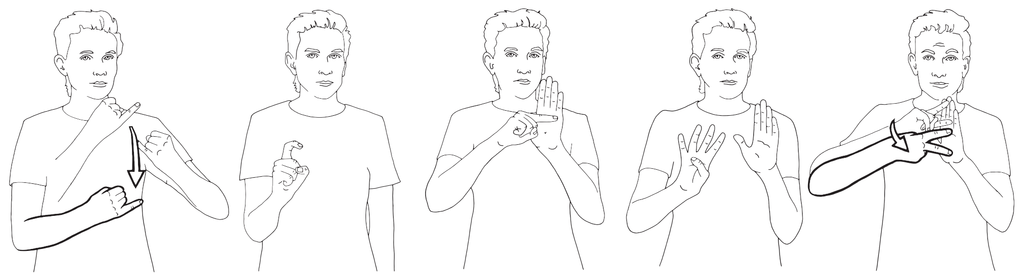

MINUS


ONE,

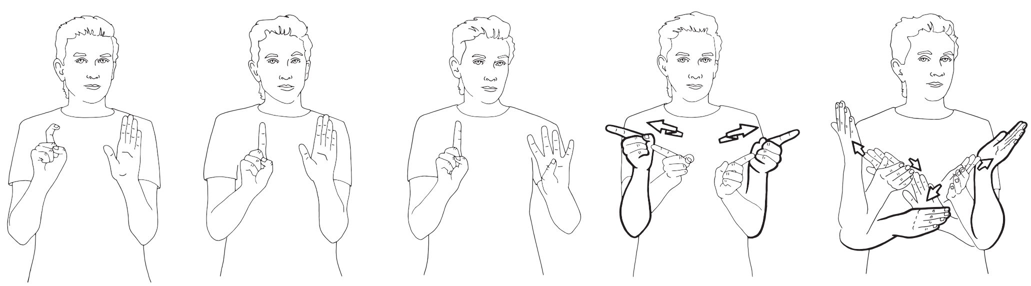

LEFT


SQUARED

EXPRESSION, REF3 EASY FACTOR. SEE



END X MINUS FOUR, MEANS



X ONE, FOUR, ONE-OF-THOSE. ALL-DONE.

There are surprisingly few standardized mathematical signs in American Sign Language. A 2005 survey of professors, teachers, interpreters, and students at the National Technical Institute for the Deaf found agreement on just eight of twenty-five basic mathematical signs (e.g., consensus on ADD but not EXPONENT).[140] Richard Ladner, a computer scientist and math PhD working on technological tools for the deaf community, explains this as follows: "American Sign Language (ASL) is very young relative to other languages, originating in the early 1800s, and only recently recognized as a language with the pioneering work of [William] Stokoe in 1960. . . . With small numbers of deaf students in advanced science being geographically dispersed, the growth of ASL has been severely inhibited in Science, Technology, Engineering, and Math (STEM) fields."[141] Without a standard sign, instructors will either finger-spell a word, use an existing sign (e.g., FUNCTION), or invent one on the spot.[142] Each option has its drawbacks. "In my opinion," writes James Nickerson, Professor and Mathematics Program Director in the Department of Science, Technology, and Mathematics at Gallaudet University, "too many mathematical signs are taken directly from the standard sign dictionary without regard for the relevant mathematical concept."[143]

This exercise was the product of many hands, not just the pair shown. Christopher Hayes, a mathematics PhD student at the University of Connecticut, translated the proof from English into ASL, and Peggy Swartzel Lott of the Department of Linguistics at the University of California San Diego, along with Daniel W. Renner and Rob Hills, illustrated the signs as modeled by Erin Oleson Dickson. The proof exhibits some of the distinctive features of sign language, including its spatial organization and use of referents.

75

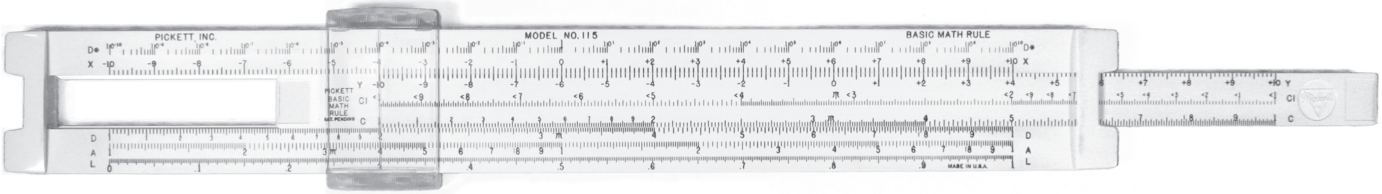
Slide Rule

Problem: Solve $x^3-6x^2+11x-6=2x-2$.

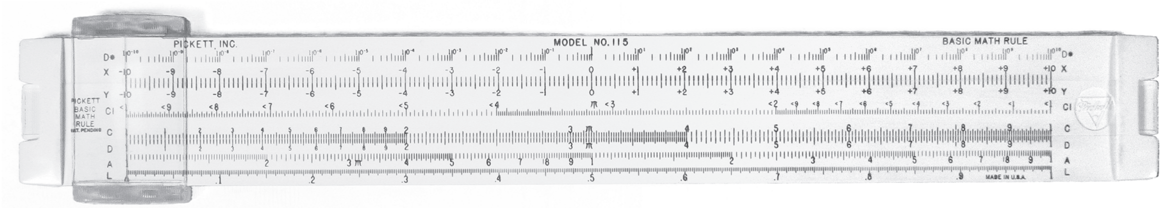
The cubic polynomial equation in standard form is $x^3-6x^2+9x-4=0$.

Substitute $y+2$ for x to obtain the reduced cubic equation $y^3-3y-2=0$.

Move the slide until the difference between the marking on the A scale under the C index and the marking on the C scale above 2 on the D scale equals 3. The reading on the D scale under the C index is the first root $y_1=2$, and the reading on C above 2 on D is the product of the remaining roots $y_2y_3=1$.



To find y_2 and y_3, move the slide until the sum of the marking on the D scale under the C index and the marking on the C scale above 1 on the D scale equals $|-y_1|$ or 2. The reading on the D scale under the C index is $-y_2=1$, and the reading on C above 1 on D is $-y_3=1$.



Thus the roots of the reduced cubic are $y_1=2$, $y_2=-1$, $y_3=-1$. Adding 2 to each of these roots yields the roots of the given equation.

Solution: $x_1=4$, $x_2=1$, $x_3=1$.

The principle underlying a slide rule's function is actually simpler than this exercise makes it sound. A movable ruler set against a fixed ruler allows one to add lengths by placing them end to end. Two such rulers marked with logarithmic scales allow one to multiply and divide, since the sum of the log of two numbers is the log of their product. Both the C scale on the slide and the D scale on the fixed rule are logarithmic scales that range over a power of ten, beginning from the index, which is the 1 marked on the left end of the rule. Setting the index of the C scale over d on the D scale, the slide rule shows the product cd on the D scale under c on the C scale.

The product of the roots of a cubic $y_1 y_2 y_3$ equals the constant coefficient. This fact is one of the three Vietà formulas for the cubic; the other two appear in 69 Statistical. Our approach to the solution with the slide rule will be to find $\pm y_1$ by reading D under the index of C and find $\pm y_2 y_3$ by reading C over $|-2|$ on D.

But first we need to fix the position of the slide, which we do using a second condition on the pair of numbers y_1 and $y_2 y_3$. Substituting the product of roots for the constant gives $y^3 - 3y - y_1 y_2 y_3 = 0$; that is, $y^3 - y_1 y_2 y_3 = 3y$. Since the roots are nonzero, we can divide this equation by $y = y_1$ to get $y_1^2 - y_2 y_3 = 3$. This is our second constraint.

Now the A scale on a slide rule is a two-decade logarithmic scale that shows the square of the marking on D. Thus, when the slide is positioned so that the difference between the marking on A over the index of C and the marking on C above $|-2|$ on the D scale equals 3, then this second condition is satisfied. We read $\pm y_1$ on the D scale beneath the C index. Examining the sign of the coefficients of the reduced cubic, it turns out that this reading is in fact equal to $+y_1$.

A similar approach yields the remaining roots y_2 and y_3. The first condition is the one just found, namely the reading equal to $\pm y_2 y_3$, and the second is that the sum $y_1 + y_2$ must equal $-y_1$, since the roots of a reduced cubic sum to zero.[144]

The slide rule pictured is the Pickett Model 115 Basic Math slide rule, and it was supplied by Ebay vendor librariesrcool. A smaller, metal rule, the Pickett Model N600-ES, accompanied NASA crew aboard the Apollo 13 mission.[145] Thanks go to Robert Dawson for suggesting this variation.

76

Experimental

To solve the equation $x^3-6x^2+11x-6=2x-2$, we employ Newton's method to find the roots of the polynomial

$$p(x)=x^3-6x^2+9x-4.$$

That is, we begin with an initial root estimate r_0, and we obtain each subsequent estimate r_{i+1} using linear approximation:

$$r_{i+1}=r_i-\frac{p(r_i)}{p'(r_i)}.$$

This root-finding scheme was implemented using a computer algebra system with initial points $r_0=0,10$. The resulting data suggest that at least two of the roots are very near if not equal to 1 and 4. (See the following table.)

i	r_i	$p(r_i)$	r_i	$p(r_i)$
0	0	−4	10	486
1	0.44444444	−1.09739369	7.42857143	141.69096210
2	0.70209340	−0.29268375	5.76958525	40.25619493
3	0.84460980	−0.07619041	4.75376676	10.62115027
4	0.92043779	−0.01949408	4.21597868	2.23376352
5	0.95971156	−0.00493487	4.02557435	0.23411012
6	0.97972319	−0.00124178	4.00042516	0.00382751
7	0.98982768	−0.00031148	4.00000012	0.00000109
8	0.99490526	−0.00007800	4.00000000	$\sim 10^{-13}$
9	0.99745047	−0.00001952	4.00000000	$\sim 10^{-27}$
10	0.99872470	−0.00000488	4.00000000	$\sim 10^{-56}$
11	0.99936221	−0.00000122	4.00000000	$\sim 10^{-113}$
12	0.99968107	−0.00000031	4.00000000	$\sim 10^{-226}$
13	0.99984053	−0.00000008	4.00000000	$\sim 10^{-453}$
14	0.99992026	−0.00000002	4.00000000	$\sim 10^{-907}$
15	0.99996013	$\sim 10^{-9}$	4.00000000	$\sim 10^{-1815}$

We applied Newton's method to one thousand additional choices of initial point r_0, sampled at random in the range $-10^9 \le r_0 \le 10^9$. The results in each case displayed the behavior of one or the other of the two patterns shown in the table.

Experimental

“Mathematics is not a deductive science—that’s a cliché,” insists Paul Halmos. “When you try to prove a theorem, you don’t just list the hypotheses, and then start to reason. What you do is trial and error, experimentation, guesswork. You want to find out what the facts are, and what you do is in that respect similar to what a laboratory technician does.”[146]

Newton’s original use of the eponymous method was not formulated explicitly in terms of the derivative but in terms of secants (the form here is due to Joseph Raphson), and in this way it can be thought of as a generalization of the method of double false position used in 34 Medieval.[147] As it happens, Newton developed the method of approximation in order to solve $y^3 - 2y - 5 = 0$, a cubic equation.[148]

Experimentation in mathematics is not impossible without a computer, and plenty of mathematicians do their guesswork the same way Newton did, with pencil and paper. But the impact of computers, especially in certain areas of research, cannot be overstated. The following 77 Monte Carlo method is a good example.

77

Monte Carlo

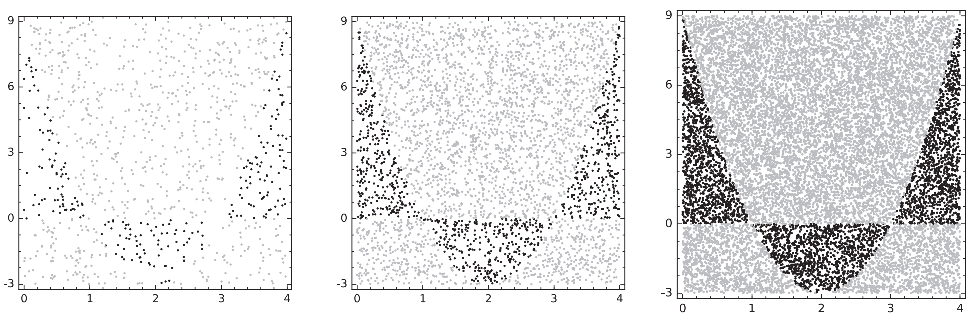
To verify that the roots of $x^3-6x^2+11x-6=2x-2$ are $x=1$ or $x=4$, we will express the roots in terms of the fundamental theorem of calculus and approximate the resulting integral equation. Subtracting $2x-6$ from both sides, we have $x^3-6x^2+9x=4$. If we define $f(x)=x^3-6x^2+9x$, then a root $x=a$ is a real number such that $f(a)=4$. According to the first fundamental theorem of calculus

$$f(a)=\int_0^a f'(x)\,dx$$

where the right-hand side is the integral of the derivative $f'(x)=3x^2-12x+9$. We can approximate the integral $f(a)$ by randomly distributing n points across the rectangle that contains the graph of $f'(x)$ on $0\le x\le 4$ and multiplying the area of the rectangle by the fraction of points that fall within the desired area. As the integral represents a signed area, points falling above the derivative f' and below the x-axis are subtracted from the number of points falling above the x-axis and below f'.

The figure shows the random distribution of $n=1\,000, 4\,000$, or $16\,000$ points over the rectangle $[0,4]\times[-3,9]$ and the resulting estimates for $f(1)$ and $f(4)$. By the central limit theorem, the error of each succeeding estimate is half the error of the estimate that precedes it. These area estimates converge to within 0.4% of 4. Direct inspection of the parabola $y=3x^2-12x+9$ strongly suggests that 1 and 4 are the only roots; in particular, $f(a)>4$ when $a>4$, and $f(a)<0$ when $a<0$.

$n=1\,000$ $\quad$ $n=4\,000$ $\quad$ $n=16\,000$



$f(1)\approx 3.2924 \quad f(4)\approx 3.6480$ $\qquad$ $f(1)\approx 4.0325 \quad f(4)\approx 3.9720$ $\qquad$ $f(1)\approx 3.9911 \quad f(4)\approx 4.0170$

Monte Carlo

A widespread approach to statistical simulation, the Monte Carlo method aims to circumvent an intractable computation by producing a large set of random sample computations which, according to probabilistic arguments, tend to closely approximate the desired solution. A typical application of this method is to compute a high-dimensional volume or integrate a nonlinear differential equation. The Polish-American mathematician and nuclear physicist Stanislaw Ulam is supposed to have hit on the idea during a game of solitaire.[149] His coauthor, Nick Metropolis, who besides having the perfect film-noir name was also a founding member of the Los Alamos National Laboratory, is said to have named the method for the world-famous casino. In their 1949 paper "The Monte Carlo method," they describe it as follows:

> The essential feature of the process is that we avoid dealing with multiple integrations or multiplications of the probability matrices, but instead sample single chains of events. We obtain a sample of the set of all such possible chains, and on it we can make a statistical study of both the genealogical properties and various distributions at a given time. . . . We want now to point out that modern computing machines are extremely well suited to perform the procedures described.[150]

This last point is critical. The Monte Carlo method is an early example of the impact of computing technology on mathematical styles of reasoning. For other examples, see 48 Computer Assisted and 97 Psychedelic.

78

Probabilistic

Suppose that $x^3 - 6x^2 + 11x - 6 = 2x - 2$ has three real solutions x_1, x_2, x_3, with mean $\overline{x}$ and standard deviation s. The likelihood of choosing the maximum $x_{\max}$ among them is at least $1/3$, with it exceeding $1/3$ if $x_{\max}$ is a multiple root. Similarly, the chance that the real number $(x_i - \overline{x})/s$ will be the maximum among the set of real numbers $\{(x_i - \overline{x})/s : i = 1, 2, 3\}$ is at least $1/3$. If X is the discrete uniform random variable $X = (x_i - \overline{x})/s$, then we may denote this probability by the inequality

$$\frac{1}{3} \leq Prob\left(X = \frac{x_{\max} - \overline{x}}{s}\right). \tag{1}$$

Note that the expected value $E(X)$ of X is 0 and its variation $s^2(X)$ is 1. Knowing this, Cantelli's inequality implies that for any real number $l > 0$,

$$Prob\,(X \geq l) \leq \frac{1}{1 + l^2}.$$

If we make $l = (x_{\max} - \overline{x})/s$, the above inequality becomes

$$Prob\left(X \geq \frac{x_{\max} - \overline{x}}{s}\right) \leq \frac{1}{1 + \left(\frac{x_{\max} - \overline{x}}{s}\right)^2}. \tag{2}$$

Since, by definition, no root x_i is greater than $x_{\max}$, the inequalities (1) and (2) combine as

$$\frac{1}{3} \leq Prob\left(X = \frac{x_{\max} - \overline{x}}{s}\right) \leq Prob\left(X \geq \frac{x_{\max} - \overline{x}}{s}\right) \leq \frac{1}{1 + \left(\frac{x_{\max} - \overline{x}}{s}\right)^2}.$$

The ordering of the first and last terms implies that $((x_{\max} - \overline{x})/s)^2 \leq 2$. Given that $x_{\max} - \overline{x} \geq 0$, it follows that $x_{\max} \leq \overline{x} + s\sqrt{2}$. A similar argument produces the lower bound $\overline{x} - s\sqrt{2} \leq x_{\min}$. Combining the inequalities, we have

$$\overline{x} - s\sqrt{2} \leq x \leq \overline{x} + s\sqrt{2}. \tag{3}$$

The Vietà formulas provide the mean and standard deviation in terms of the coefficients a_1, a_2 of the linear and quadratic terms as follows: $\overline{x} = -a_2/3 = 2$ and $s = \sqrt{(2a_2^2 - 6a_1)/9} = \sqrt{2}$. Substituting the values of $\overline{x}$ and s into the inequality (3), we obtain the following bounds on the supposed three real roots:

$$0 \leq x \leq 4.$$

This variation essentially proves the Laguerre-Samuelson inequality

$$|x| \leq \overline{x} \pm s\sqrt{n-1}$$

for cubic ($n=3$) polynomials with all real roots. In this way, it is very similar to 69 Statistical, and that variation explains in detail how Vietà's formulas produce the mean and standard deviation in the last paragraph of this variation. The probabilistic argument here is due to William P. Smith.[151]

Cantelli's inequality is

$$Prob\left(X-\overline{x}\geq l\right)\begin{cases}\leq & \frac{s^2}{s^2+l^2} & \text{if } l>0,\\ \geq & 1-\frac{s^2}{s^2+l^2} & \text{if } l<0.\end{cases}$$

I wasn't able to uncover anything about William P. Smith, but Francesco Paolo Cantelli was a twentieth century Italian mathematician and actuary who also studied astronomy at the Observatory in Palermo. Early in his career, he dated a celestial configuration that appeared in the night sky over Italy 600 years prior and is described in Dante's *Divine Comedy*.[152]

79

Intuitionist

Theorem. *If n is a natural number and $n^3-6n^2+11n-6=2n-2$, then $n=1$ or $n=4$.*

Proof. We seek a constructive way of finding for each natural number $n=k$ either a proof that $k^3-6k^2+11k-6=2k-2$ implies $k=1$ or a proof that it implies $k=4$.

Effecting the mental constructions indicated by $k^3-6k^2+11k-6k$ and $2k-2$ for $k=1,2,3,4$, and 5, the creating subject finds that they lead to the same result when $k=1$ and 4 and lead to different results when $k=2,3$, and 5.

For any k,

$$\begin{aligned} k^3-6k^2+11k-6 &= 2k-2+(k^3-6k^2+9k-4) \\ &= 2k-2+[k^2(k-6)+8k+(k-4)]. \end{aligned}$$

If $k\geq 6$, then each term within the square brackets—and hence their sum—is strictly positive. Thus the quantity k^3-6k^2+11k is strictly greater than $2k-2$. In this way, we have a constructive method to refute the hypothesis $k^3-6k^2+11k-6=2k-2$, and therefore the implication is vacuously true for each $k\geq 6$.

Intuitionist

The mathematical philosophy known as intuitionism arose as one of the alternatives to Platonism, the belief that mathematical objects exist in some form of reality, during the foundational crisis in mathematics at the beginning of the twentieth century. Its founder, Dutch mathematician L. E. J. Brouwer, conceived of mathematics as the mental constructions of a creating subject dependent on time.[153] Thus, for example, his student Arend Heyting explains, "A mathematical theorem expresses a purely empirical fact, namely the success of a certain construction. '$2+2=3+1$' must be read as an abbreviation for the statement: 'I have effected the mental constructions indicated by '$2+2$' and by '$3+1$' and I have found that they lead to the same result."[154] That this proof is restricted to the natural numbers is a reflection of my limited knowledge of intuitionism, not its proper scope. A full-blooded intuitionist solution of this cubic over the continuum ought to exhibit the "choice sequences" that characterize the intuitionistic real numbers.[155]

80

Paranoid

a a a a a a a a a a a a a a a a a a a a a a a a a a a a a a a a a a a b b b b b b b c c c c c c c
c c c c c c d d d d d d d d d d d d d d d e e e e e e e e e e e e e e e e e e e e e e e e e e e
e e e e e e e e e e e e e e e e e e e e f f f f f f f f f f f f g g g g g g g g g h h h h h h h h h
h h h h h i i i i i i i i i i i i i i i i i i i i i i i i i i i k k l l l l l l l l l l l l l l l m m m m m m
m m m m m m m m n n n n n n n n n n n n n n n n n n n n n n n n n n o o o o o o o o
o o o o o o o o o o o o o o o o o o o o o o o o o p p p p p p p p q q q q q q r r r r r r r r r
r r r r r r r r r r r r r r r r s s s s s s s s s s s s s s s s s s s s s s s s s s s s s s s s s s s t t t t t
t t t t t t t t t t t t t t t t t t t t t t t t t t t t t t t t t t t t t t u u u u u u u u u u u u u u u u u u u
u u v v w w w w w w x x x x x x x y y y y y y

This proof contains all the letters used by another exercise in this book (which one?) arranged in alphabetical order.

Galileo used anagrams as a way to ensure priority of a discovery at the outset, thus affording him time to verify his results before claiming them more publicly. For example, observing Saturn in 1610, he sent Kepler the jumble of letters

smaismrmilmepoetaleumibunenugttauiras

that could be rearranged into the Latin for "I observed the highest planet to be triple-bodied."[156] With more powerful telescopes, the extra bodies about Saturn were revealed to be its rings. The Dutch astronomer and mathematician Christiaan Huygens made this discovery in 1656 and published it in the form of an anagram also, this time arranging the letters in alphabetic order:

aaaaaaacccccdeeeeehiiiiiiillllmmnnnnnnnnnooooppqrrstttttuuuuu.[157]

A famously secretive mathematician, Isaac Newton encoded his discovery of the calculus when describing one of its applications in a 1676 letter to the German theologian and natural philosopher Henry Oldenburg.[158] In his shorthand, coefficients count the more frequently appearing letters,

> At present I have thought fit to register [my methods] by transposed letters, lest through others obtaining the same result, I should be compelled to change the plan in some respects.
>
> 5accdæ10effh11i4l3m9n6oqqr8s11t9y3x:
> 11ab3cdd10eæg10ill4m7n6o3p3q6r5s11t8vx,
> 3acæ4egh5i4l4m5n8oq4r3s6t4v, aaddæcecceiijmmnnooprrrsssssttuu.

This little history of competition and paranoia left me wondering if these weren't unavoidable features of modern mathematical practice. While this could be true, they may just signal periods of mathematical growth, modern or not. Apparently Archimedes planted false theorems in his mathematical letters so as to defend himself against lesser-minded plagiarists.[159]

81

Doggerel

Suppose x cub'd less four were equal to
Six times x squar'd less nine times x. Before
Deducing more, let's state up front the two
Solutions: x is one, or x is four.
Potentially, how might this manifest?
What's given can be made quadratic free,
(i.e., by change of character), depress'd:
y cub'd amounts to two and y times three.
Again, a clever transformation: add
To that the square of y on left and right.
Let not the complication make us sad;
A simple factoring puts all in sight.
Thus y is two or minus one, and more,
Back substituting, x is one or four.

There is a long history of writing mathematics in verse, as both an aid to memory and an encomium to the mathematics it expresses. Much of medieval Indian mathematics, including the Līlāvatī by Bhāskara II, was composed in Sanskrit verse.[160] See the note after 68 Word Problem for an excerpt of one translator's attempt to capture the rhythm of this text.

Tartaglia committed his solution of the cubic equation $x^3 + cx = d$ to memory with a twenty-five line verse. Following the *terza rima* form that Dante Alighieri created for his *Divina Commedia*, Tartaglia's poem begins

Quando che'l cubo con le cose appresso
Se agguaglia à qualche numero discreto
Trouan dui altri differenti in esso.[161]

For a translation of Tartaglia's poem that preserves the interlocking three-line rhyme scheme, see "*Quando Che'l Cubo*" by Kellie Gutman of West Roxbury, Mass.[162]

A minor note on the sonnet here: the word *depressed* in the seventh line is actually a technical term used to describe a cubic equation without a quadratic term. As mentioned in the comment on 29 Model, every cubic can be depressed by making a change of variable (or "character") under a so-called Tschirnhaus transformation, in this case, $x = y + 2$.

After writing these lines, I came across the following stanza from Lewis Carroll's double acrostic, *The First Riddle*.[163] It's also in iambic pentameter, assuming you can figure out how to scan an equation.

Yet what are all such gaieties to me
Whose thoughts are full of indices and surds?

$$x^2 + 7x + 53$$
$$= \frac{11}{3}.$$

82

Inconsistency

Corollary. *Let $x \in \mathbb{R}$. If $x^3 - 6x^2 + 11x - 6 = 2x - 2$, then $x = 1$ or $x = 4$.*

Proof. Suppose arithmetic is inconsistent, then there exists a proposition P that is both true and false. Since P is true, P or the equation $x = 1$ is true. Since P is also false, the fact that at least one of P or $x = 1$ is true implies that $x = 1$ must be true. By a similar argument, $x = 4$ is true. □

As some politicians know well, contradiction is a very effective method of argumentation. In logic, the notion that any proposition follows from (or is a corollary to) a contradiction is known as the principle of explosion. Indeed, this exercise deduces, without need of any additional premise, that x is *both* 1 and 4. The proof makes use of two basic logical transformations: disjunction introduction (since P is true, P or $x=1$ is true) and disjunctive syllogism (P is also false…implies that $x=1$ must be true).

Avoiding the question of whether or not arithmetic is consistent—a question that, at this time, is not entirely resolved—it is possible to devise inconsistent logical systems that are not trivial in the sense that not every statement is true. Such logics are called paraconsistent, and they have their origin in the "imaginary" logic of Nicolai A. Vasiliev.[164]

For a critique of this proof, see 83 Correspondence.

83

Correspondence

Date: Sat, 12 Aug 2017 10:51:40 -0700 (PDT)
From: Roman <RKossak@gc.cuny.edu>
To: me <pording@slc.edu>
Subject: 82 Inconsistency

There are problems with 82. You say "Suppose arithmetic is inconsistent." To make sense of it we need to know what is arithmetic, and what it means to be inconsistent. And then, the most difficult of all, one also needs to know what is truth.

In mathematical logic, consistency is a property of a formal system that includes axioms and rules of proof. A system is consistent, if in it one cannot derive a statement and its negation. There is nothing here about truth. It is all about derivations of formal consequences from formal assumptions using formal rules of proof.

In classical logic, according to its rules of proof, one can derive any sentence from P & not P. Hence, in an inconsistent system one can derive any sentence (written in the formal language of the system).

If one day we discovered that, for instance, Peano Arithmetic is inconsistent (as a first-order system), we would not declare that from now on we know that every sentence about numbers is true. Instead, we would examine the axioms, or the system (or both), and we would try to fix them to eliminate the inconsistency.

In other words, there is no direct link between inconsistency and truth, first of all because the former is a well defined formal concept, and the latter is intuitive and vague, but also even if one formalizes truth, for example via Tarski's definition, one still cannot say for sure that what has been derived is "true," because systems that we strongly believe to be consistent can prove false arithmetic statements (they can even prove that some unsolvable polynomial equations have solutions).

The question whether arithmetic is consistent is resolved in the same sense as the one in which we believe that it is not possible that today is Sunday, and it is not Sunday. What some sophisticated mathematicians (including Voevodsky and Ed Nelson) question, is whether Peano Arithmetic is consistent, in the sense I described above.

It all is something it would be a pleasure to ponder on a beach under a full moon.

r

Correspondence

My friend Roman Kossak, a logician at the City University of New York, sent this e-mail after reviewing a draft of the previous exercise 82 Inconsistency. E-mail is as important to collaboration in mathematics as any other art or science. Since the advent of $\TeX$ there's even a convention for transmitting mathematical notation using a regular keyboard (see 35 Typeset). For some high-level research examples, see Cédric Villani's *Birth of a Theorem: A Mathematical Adventure*, in which the French mathematician reproduces numerous e-mails verbatim from his odyssey to capture the Fields Medal. (Then again, his English translator calls the book a "work of literary imagination," so maybe Villani's correspondence is completely fabricated.[165]) As in Villani's book, I've styled the look of this letter after the console-based e-mail client of the sort (e.g., Pine) preferred by some mathematicians for its speed, customizability, and simplicity.

p^3/q^2	x_1	x_2	x_3
−6.95	0.8871209	1.0867409	4.0261382
−6.94	0.8903176	1.0848419	4.0248406
−6.93	0.8935810	1.0828769	4.0235420
−6.92	0.8969170	1.0808405	4.0222425
−6.91	0.9003320	1.0787260	4.0209420
−6.90	0.9038335	1.0765259	4.0196406
−6.89	0.9074304	1.0742315	4.0183381
−6.88	0.9111331	1.0718322	4.0170347
−6.87	0.9149541	1.0693157	4.0157303
−6.86	0.9189084	1.0666667	4.0144249
−6.85	0.9230149	1.0638666	4.0131185
−6.84	0.9272970	1.0608918	4.0118112
−6.83	0.9317850	1.0577122	4.0105028
−6.82	0.9365190	1.0542876	4.0091935
−6.81	0.9415540	1.0505629	4.0078831
−6.80	0.9469694	1.0464588	4.0065718
−6.79	0.9528870	1.0418536	4.0052594
−6.78	0.9595117	1.0365423	4.0039461
−6.77	0.9672434	1.0301249	4.0026317
−6.76	0.9771147	1.0215689	4.0013164
−6.75	1.0000000	1.0000000	4.0000000
−6.74	$1.0006587-0.0222173i$	$1.0006587+0.0222173i$	3.9986826
−6.73	$1.0013179-0.0314132i$	$1.0013179+0.0314132i$	3.9973642
−6.72	$1.0019776-0.0384646i$	$1.0019776+0.0384646i$	3.9960448
−6.71	$1.0026378-0.0444053i$	$1.0026378+0.0444053i$	3.9947244
−6.70	$1.0032985-0.0496356i$	$1.0032985+0.0496356i$	3.9934029
−6.69	$1.0039598-0.0543611i$	$1.0039598+0.0543611i$	3.9920805
−6.68	$1.0046215-0.0587036i$	$1.0046215+0.0587036i$	3.9907570
−6.67	$1.0052838-0.0627428i$	$1.0052837+0.0627428i$	3.9894325
−6.66	$1.0059466-0.0665340i$	$1.0059466+0.0665340i$	3.9881069
−6.65	$1.0066098-0.0701173i$	$1.0066098+0.0701173i$	3.9867803
−6.64	$1.0072737-0.0735232i$	$1.0072737+0.0735232i$	3.9854527
−6.63	$1.0079380-0.0767753i$	$1.0079380+0.0767753i$	3.9841240
−6.62	$1.0086028-0.0798923i$	$1.0086028+0.0798923i$	3.9827943
−6.61	$1.0092682-0.0828895i$	$1.0092682+0.0828895i$	3.9814636
−6.60	$1.0099341-0.0857794i$	$1.0099341+0.0857794i$	3.9801318
−6.59	$1.0106005-0.0885726i$	$1.0106005+0.0885726i$	3.9787990
−6.58	$1.0112675-0.0912778i$	$1.0112675+0.0912778i$	3.9774651
−6.57	$1.0119350-0.0939028i$	$1.0119350+0.0939028i$	3.9761301
−6.56	$1.0126029-0.0964540i$	$1.0126029+0.0964540i$	3.9747941

The oldest extant mathematical tables date from 2600 BCE and predate the earliest known mathematical texts by seven centuries (see 16 Ancient).[166] Numerical tables for mathematical and other kinds of calculation (e.g., taxes) are still published today despite the widespread availability of much more powerful computers in the form of smartphones. Looking up a solution by hand, locating it among its neighbors, gives you confidence that you've found the right number. Perhaps too much confidence—how do we know whether $x_1 = 1.0000000$ means $x_1 = 1$ or $x_1 = 1.00000003$?

H. A. Nogrady discovered that after making the usual substitution to reduce a cubic equation to the square-free form $y^3 + py + q = 0$ (see 29 Model), a second change of variable, $y = qz/p$, puts the cubic into the form[167]

$$z^3 + \frac{p^3}{q^2} z + \frac{p^3}{q^2} = 0.$$

I find this remarkable. Every cubic is characterized by essentially one parameter p^3/q^2, and we can just list them out in the form of a table.[168] Our cubic equation $x^3 - 6x^2 + 11x - 6 = 2x - 2$ reduces to $y^3 - 3y - 2 = 0$, and its solutions appear in the row $p^3/q^2 = (-3)^3/(-2)^2 = -6.75$, right at the precipice of the complex-valued solutions.

85

Exhaustion

Theorem. *Let n be an integer. If $n^3 - 6n^2 + 11n - 6 = 2n - 2$, then n is 1 or 4.*

Proof. The difference between the terms in the unknown integer n on each side of the equation equals the difference between the constant terms on each side of the equation

$$(n^3 - 6n^2 + 11n) - (2n) = (-2) - (-6).$$

As n divides the difference of the variable terms, so it must also divide the difference of the constant terms. But the difference of the constants is 4, which, being integral, admits only finitely many divisors: $\{-4, -2, -1, 1, 2, 4\}$. We identify the solutions of the equation by testing each divisor d in this set to see if it satisfies the equation.

Case 1. If $d = -4$, then $d^3 - 6d^2 + 11d - 6 = -210$. Yet $2d - 2 = -10$, hence $n \neq -4$.

Case 2. If $d = -2$, then $d^3 - 6d^2 + 11d - 6 = -60$. Yet $2d - 2 = -6$, hence $n \neq -2$.

Case 3. If $d = -1$, then $d^3 - 6d^2 + 11d - 6 = -24$. Yet $2d - 2 = -4$, hence $n \neq -1$.

Case 4. If $d = 1$, then $d^3 - 6d^2 + 11d - 6 = 0$, and $2d - 2 = 0$. Thus $n = 1$ is a solution.

Case 5. If $d = 2$, then $d^3 - 6d^2 + 11d - 6 = 0$. Yet $2d - 2 = 2$, hence $n \neq 2$.

Case 6. If $d = 4$, then $d^3 - 6d^2 + 11d - 6 = 6$, and $2d - 2 = 6$. Thus $n = 4$ is a solution.

Thus the solution is $n = 1$ or $n = 4$, as claimed. □

Exhaustion

A proof by exhaustion divides a proposition into a finite list of cases, shows that these cases exhaust all possible situations in which the proposition may hold, and then proceeds by checking each case individually. In this proof, there are six cases corresponding to the six divisors of the constant term of the cubic polynomial. A similar proof appears in 73 Another Translated. It isn't very exhausting to check six (simple) cases, but this style of proof may require a much larger number of cases, each of which may involve more complicated computation.

Proofs by exhaustion (not to be confused with the infinite *method of exhaustion*) are sometimes amenable to computer assistance, for better or worse. For better, because the automation of case-checking accelerates the work; for worse, because such expediency makes for unenlightening proofs. For this reason, proofs by exhaustion are sometimes referred to as "brute force." Not surprisingly, this is the form taken by proofs that hold the record for being the longest ever produced. For example, in 2016, Marijn Heule, Oliver Kullmann, and Victor Marek presented a 200-terabyte proof of the Boolean Pythagorean triples problem that involves verifying nearly a trillion cases.[169]

86

Another Substitution

Notations

Zero and *one* are styles, and they are denoted by 0 and 1 respectively. The *synthesis* of styles x and y is the result of combining x and y, and it is denoted by $x+y$; their *intersection* is the result of crossing x and y, and it is denoted by $x\times y$ or xy. Styles x and y are *equivalent* if they are indistinguishable, a fact which is denoted by the equivalence $x\sim y$.

Definitions

1. The styles 2 through 11 are defined by the syntheses $2\sim 1+1$, $3\sim 2+1$, $\ldots$, $11\sim 10+1$.
2. The *antithesis* of a style x is the style $-x$ such that $x+(-x)\sim 0$.
3. The *difference* of two styles x and y is denoted $x-y$, and it is defined as the synthesis $x+(-y)$.
4. The *square* of a style x is the the intersection of x with itself, denoted by x^2.
5. The *cube* of a style x is the intersection of x with its square, denoted by x^3.

Axioms

6. Given any proposition P, if P or P, then P.
7. For all styles x and y, if $x\sim y$, then $y\sim x$.
8. For all styles x, y, z, if x is equivalent to y and y is equivalent to z, then x is equivalent to z.
9. For all styles x and y, and any equivalence E, if x is equivalent to y, then y may be substituted for any occurrence of x in E without changing the truth value of E.
10. If x and y are styles, then the synthesis $x+y$ and intersection $x\times y$ are styles also.
11. For all styles x, y, z, if x is equivalent to y, then the syntheses $x+z$ and $y+z$ are equivalent, as are the intersections $x\times z$ and $y\times z$.
12. For all styles x, y, the transposed syntheses $x+y$ and $y+x$ are equivalent, as are the transposed intersections $x\times y$ and $y\times x$.
13. For all styles x, y, z, the triple syntheses $(x+y)+z$ and $x+(y+z)$ are equivalent, as are the triple intersections $(x\times y)\times z$ and $x\times(y\times z)$.
14. If x, y, z are styles, then the intersection of x with the synthesis $y+z$ is equivalent to the synthesis of the intersections $x\times y+x\times z$.
15. The style 1 is not equivalent to the style 0.
16. For any style x, the synthesis $0+x$ is equivalent to x.
17. For any style x, the intersection $1\times x$ is equivalent to x.
18. For any style x, there exists a unique antithesis $-x$.
19. For any styles x, y, if $x\times y\sim 0$, then $x\sim 0$ or $y\sim 0$.

Theorems

20. For all styles x, y, z, if $x \sim y$, then $x - z \sim y - z$.
21. For any style x, $x - x \sim 0$.
22. For any style x, $0 \times x \sim 0$.
23. For any styles x, y, $(-x)y \sim -(xy) \sim x(-y)$
24. For any style x, $-(-x) \sim x$
25. For any styles x, y, z, $x(y - z) \sim xy - xz \sim (y - z)x$
26. For any styles x, y, z, w, $(x - y)(z - w) \sim xz - xw - yz + yw$
27. For any style x, $x + x \sim 2x$
28. For any styles x, y, $-(x + y) \sim -x - y$
29. $-2 + (-4) \sim -6$
30. $1 + 4 \times 2 \sim 9$
31. For any style x, $(x - 1)^2 \sim x^2 - 2x + 1$.
32. For any style x, $(x - 1)^2(x - 4) \sim x^3 - 6x^2 + 9x - 4$.
33. For any style x, $x^3 - 6x^2 + 9x - 4 \sim (x^3 - 6x^2 + 11x - 6) - (2x - 2)$.
34. For any style x, if $x^3 - 6x^2 + 11x - 6 \sim 2x - 2$, then $x \sim 1$ or $x \sim 4$.

PROOF. Suppose x is a style.

Theorem 33	$x^3 - 6x^2 + 9x - 4 \sim (x^3 - 6x^2 + 11x - 6) - (2x - 2)$	(1)
Hypothesis	$x^3 - 6x^2 + 11x - 6 \sim 2x - 2$	(2)
Axiom 10	$2x - 2$ is a style	(3)
Axiom 9, (1), (2), (3)	$x^3 - 6x^2 + 9x - 4 \sim (2x - 2) - (2x - 2)$	(4)
Theorem 21, (3)	$(2x - 2) - (2x - 2) \sim 0$	(5)
Axiom 8, (4), (5)	$x^3 - 6x^2 + 9x - 4 \sim 0$	(6)
Theorem 32	$(x - 1)^2(x - 4) \sim x^3 - 6x^2 + 9x - 4$	(7)
Axiom 8, (7), (6)	$(x - 1)^2(x - 4) \sim 0$	(8)
Axiom 19, (8)	$(x - 1)^2 \sim 0$ or $x - 4 \sim 0$	(9)
Definition 4, (9)	$(x - 1)(x - 1) \sim 0$ or $x - 4 \sim 0$	(10)
Axiom 19, (10)	$x - 1 \sim 0$ or $x - 1 \sim 0$ or $x - 4 \sim 0$	(11)
Axiom 6, (11)	$x - 1 \sim 0$ or $x - 4 \sim 0$	(12)
Definition 3, (12)	$x + (-1) \sim 0$ or $x + (-4) \sim 0$	(13)
Axiom 11, (13)	$x + (-1) + 1 \sim 0 + 1$ or $x + (-4) + 4 \sim 0 + 4$	(14)
Definition 2, (14)	$x + 0 \sim 0 + 1$ or $x + 0 \sim 0 + 4$	(15)
Axiom 16, (15)	$x \sim 1$ or $x \sim 4$	(Theorem)

In mathematical folklore, there's a story that David Hilbert was so committed to the formalist view of the discipline as to claim, "One must be able to say 'tables, chairs, beer-mugs' each time in place of 'points, lines, planes.'"[170] Hilbert supposedly made the remark after attending a lecture on projective plane geometry. There is a basic principle in projective plane geometry called *duality* by which one may swap the words "point" and "line" (and any corresponding terms, such as "colinear" and "concurrent") in any theorem without changing the truth of the theorem.

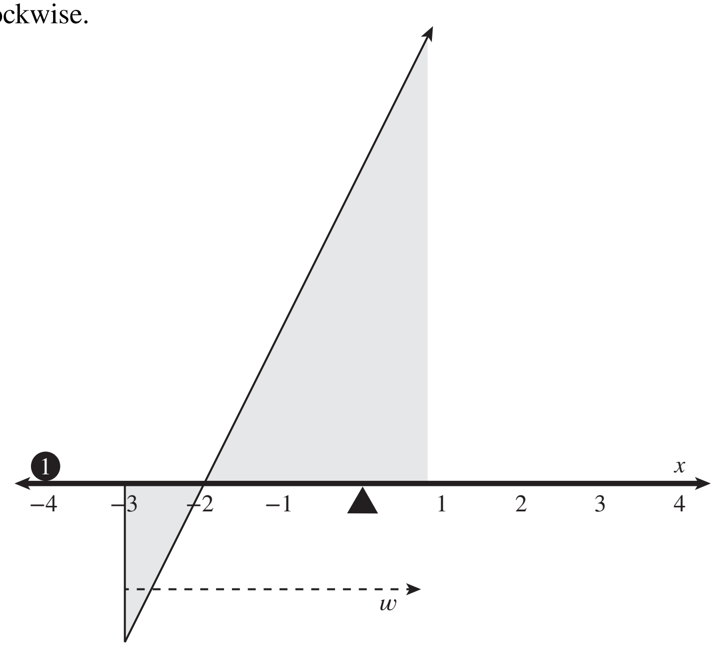
Taken out of context, the above remark could form the basis of an avant-garde manifesto. In fact, Raymond Queneau honored Hilbert's formalist dictum with the text *Les fondements de la littérature d'après David Hilbert*, in which every instance of the words "points," "lines," and "plane" in Hilbert's *Foundations of Geometry* is replaced by, respectively, "word," "sentences," and "paragraph."[171]

Working from 6 Axiomatic, I have aimed for a similar correspondence in the choice of substitutions—using "synthesis" in place of "sum" for example.

87

Mechanical

To solve the equation $w^3-6w^2+11w-6=2w-2$, first isolate the constant term to one side of the equation as follows: $4=w^3-6w^2+9w$. Let x denote the position of a point on a horizontal balance beam relative to its fulcrum at $x=0$ and measured from left to right. The law of the lever states that an object placed on a beam will exert a torque equal to the product of its mass times its distance from the fulcrum, and we adopt the convention that positive torque turns clockwise and negative torque counterclockwise.



Place a unit mass on the beam at $x=-4$; it exerts a torque of magnitude 4 units in the counterclockwise direction and represents the left side of the equation.

To represent the right side of the equation, suppose the region between the line $y=3x+6$ and the x-axis and to the right of $x=-3$ were cut from a sheet of material of width w. If the material has unit density, then a vertical slice at position x has mass equal to its height, $3x+6$, and it exerts a torque of $x(3x+6)$. Extend the width w of the region until it exactly balances the unit mass at $x=-4$, and the equation is solved.

To see why, note that the beam is in equilibrium when the region exerts a torque equal to and opposite that of the unit mass. That is, the integral of the torque of each slice of the region from its left endpoint at $x=-3$ to its right endpoint at $x=w-3$ satisfies the equation

$$4=\int_{-3}^{w-3} x(3x+6)\,dx=(w-3)^3+3(w-3)^2-(-27)-27=w^3-6w^2+9w.$$

Thus the equilibrium width w satisfies the given equation and provides the solution required.

This proof is inspired by *The* Method *of Archimedes*, also called *The Method of Mechanical Theorems*.[172] Archimedes used this to calculate the area beneath a parabola by interpreting the product of two spatial dimensions—length times width—as the product of a force with a spatial dimension—gravity times distance—according to the law of the lever. The connection to our problem is that the torque of the region of width w here is equal to the *area* under the parabola $y = 3t^2 - 12t + 9$ from $t = 0$ to $t = w$; that is, $w^3 - 6w^2 + 9w$. Without a physical balance, it is still possible to determine w geometrically by computing the region's center of mass. This is what makes Archimedes' method so effective. (For additional mechanical methods, see Frame, "Machines for Solving Algebraic Equations.")

Archimedes solved a number of other geometry problems with this method, including the computation of the volume of a sphere (balanced by a cone and cylinder!). Many calculus instructors try to motivate integral calculus by asking students to estimate this or that area or volume. Rather than telling students that we need calculus to measure these shapes it might be more honest to say, look, integrals mean we don't all have to be as clever as Archimedes.

DISCIPLE: Dear master, I seek knowledge of an unknown quantity. It is said that from your genius the desired solutions emerged.

88

Dialogue

MASTER: My dear one, I am no genius. Nevertheless, if you are willing to apply yourself, I will gladly accompany you in finding the way yourself.

DISCIPLE: The horizon of my knowledge may include linear and, perhaps, certain quadratic equations, but the cubic $x^3-6x^2+11x-6=2x-2$ is a constellation that I can but dimly glimpse in the heavens. You have too much confidence in me.

MASTER: What would you do if this were a linear equation?

DISCIPLE: I would gather like terms on one side: $x^3-6x^2+9x-4=0$. This looks simpler, but it is not quadratic.

MASTER: Indeed not, but suppose it were. Can you solve $x^2+9x-4=0$?

DISCIPLE: I would try to complete the square. If we replace x by the expression y less half the linear coefficient or $y-\frac{9}{2}$, then the linear term vanishes. The roots of the new quadratic are easily extracted, and translating these back into terms of x gives us the solutions.

MASTER: Very good. How would you guess this to go in degree three?

DISCIPLE: I have no idea. Is there such a thing as "completing the *cube*"?

MASTER: If there were, what should it be?

DISCIPLE: This is too simple-minded, but suppose I make the same change of variable as before only this time instead of taking y less half the linear coefficient, I take y less a *third* the *quadratic* term.

MASTER: Wisdom is a state of simple-mindedness.

DISCIPLE: That would mean $x=y+2$ for our cubic. Please allow me to sit. I have $y^3-3y-2=0$.

MASTER: Correct.

DISCIPLE: Now I wish to take cube roots. But there is a linear term in the way. This won't work and I am defeated.

MASTER: You are making progress, but we should not expect the path to be straight. What is it that you seek?

DISCIPLE: The roots of the cubic polynomial.

MASTER: And what do you know about quadratic roots? What do they look like? How do they behave?

DISCIPLE: They come in pairs $-b\pm\sqrt{b^2\dots}$ but I am embarrassed that I cannot remember the proper formula. Please excuse me and I will look it up.

MASTER: Do you believe what you read in books? Come, let us work with your partial formula. There are two roots $r_1=u+v$ and $r_2=u-v$, where u, v depend somehow on the coefficients of the quadratic.

DISCIPLE: Yes, and now I remember that the sum of the roots r_1+r_2 equals minus the linear coefficient of the trinomial, while their product r_1r_2 is the constant term. You do not believe that I can generalize this, do you?

MASTER: Simple mind.

DISCIPLE: We desire *three* roots. Their sum is minus the *quadratic* coefficient and their product is the constant. Is that so?

MASTER: You have your conjectures, now test them.

DISCIPLE: If I multiply three binomials I see that my guess was almost right. The sum of the roots is minus the quadratic, however their product is *minus* the constant. The linear coefficient turns out to be the sum of all products of pairs of roots. I see.

MASTER: If so, what are the roots?

DISCIPLE: The quadratic roots appeared in terms of two coefficient functions u, v, so now I had better include a third coefficient function w. But $u \pm v \pm w$ yields four roots. Something is wrong.

MASTER: What significance does ± 1 have for quadratics?

DISCIPLE: The square roots of 1 are $+1$ and -1. What we want are *cube* roots of 1. What are cube roots of 1?

MASTER: What *should* they be?

DISCIPLE: There should be three of them, and the third power of any one should equal 1. Obviously, there is 1. In addition to the root 1, let us call a second root r. If $r^3 = 1$, then surely $r^6 = 1$. But $r^6 = (r^2)^3$. Now you have made it clear to me that r^2 is a third cube root, which satisfies the identity $1 + r + r^2 = 0$. The roots of our cubic combine as:

$$r_1 = u + v + w, \qquad r_2 = u + rv + r^2 w, \qquad r_3 = u + r^2 v + rw.$$

MASTER: Good. How do these roots behave?

DISCIPLE: They must add to the coefficient of the quadratic, which is 0 in the case of our cubic. According to the identity for cube roots of unity, we have $u = 0$.

MASTER: Good. What else?

DISCIPLE: The product of all three roots must be 2, and the sum of the products of pairs must be -3. I can factor out $1 + r + r^2$, which reduces everything to two equations:

$$v^3 + w^3 = 2$$
$$vw = 1.$$

MASTER: Take one at a time.

DISCIPLE: Solving the second equation gives $w = \frac{1}{v}$. Then substituting this into the first equation gives

$$v^3 + \frac{1}{v^3} = 2.$$

Is this truly an equation of degree *six*?

$$v^6 + 1 = 2v^3.$$

MASTER: Have courage.

DISCIPLE: If it were linear, then I would collect terms

$$v^6 - 2v^3 + 1 = 0.$$

MASTER: And, if it were quadratic?

DISCIPLE: In terms of v^3, it *is* quadratic. It has a repeated root of 1, so v^3 equals 1, which means v is 1. Now we can trace our steps back to find w, y, and finally x. Thus, $w = \frac{1}{v} = 1$, so $y = 2$ or $y = r + r^2 = -1$, by the cube root of 1 identity. Finally, since $x = y + 2$, we get $x = 1$ or $x = 4$, just as you said.

MASTER: I never said that.

DISCIPLE: No, of course, you would not take credit. It is not genius.

MASTER: You misunderstand. I take credit for those falsehoods which escape my lips and my lips alone.

DISCIPLE: I must extract the other two roots r and r^2 from the equation $v^3 = 1$.

MASTER: You have begun to make your way, it is time for me to make mine.

This investigation follows the path that Timothy Gowers describes in his blog post "Discovering a formula for the cubic."[173]

Given its potential to predate written language, dialogue seems a likely candidate for the oldest style of presenting a mathematical argument. *The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven* (*Zhou Bi Suan Jing*), one of the earliest mathematical texts from China, is a dialogue that begins:

> Long ago, the Duke of Zhou asked Shang Gao "I have heard, sir, that you excel in numbers. May I ask how Bao Xi laid out the successive degrees of the circumference of heaven in ancient times? Heaven cannot be scaled like a staircase, and earth cannot be measured out with a foot rule. Where do the numbers come from?"[174]

I ought to clarify that while the dialogue of this variation and the 89 Interior Monologue that follows include some false starts, neither is an attempt to convey the actual frustration that accompanies mathematical work. That frustration is probably conveyed more directly by trying to work through whichever proof in this book the reader finds especially alienating.

That there might be a benefit to such an exercise is the premise of Carl Lindholm's book *Mathematics Made Difficult*, which opens with a very different sort of master and disciple story. Lindholm recounts a Zen koan in which the master crushes a disciple's limb in a gate. "Just as the fractured leg confused the Zen disciple," Lindholm explains, "it is hoped that this book may help to confuse some uninitiated reader and so put him on the road to enlightenment, limping along to mathematical *satori*."[175] Even if enlightenment isn't forthcoming, such exercises do help convey the weirdness of mathematics. "In mathematics you don't understand things," the Hungarian-American polymath John von Neumann famously said, "you just get used to them."[176]

89

Interior Monologue

What needs to be proved? I need to prove that x is 1 or 4.

What are the givens? I'm assuming that x satisfies $x^3-6x^2+11x-6=2x-2$.

Anything else? Well, x is a real number.

Can I test the claim for a simple example? Sure. If $x=1$, then $1-6+11-6=0$ on the left, which is equal to $2-2$ on the right. But, that's actually the converse of what I'm supposed to prove. Hm, not sure.

Can I make a picture? I have no idea. Let's move on.

How about introducing some notation? We could set $f(x)=x^3-6x^2+11x-6$ and then $g(x)=2x-2$. I'm not sure that helps, really.

Have I seen this kind of problem before? Not really.

Have I seen anything like it, possibly in a different form? I've seen individual polynomials like f and g. Subtracting g from both sides of the equation, I get $x^3-6x^2+9x-4=0$. So, what I'm really after are the roots of this polynomial.

Can I make a picture? Alright, I'll try to sketch the graph of $y=x^3-6x^2+9x-4$. Now I can see how 1 and 4 are plausible solutions.

I've used the graph of a function to find its roots before. Can I apply the same method? Yeah, but I already know the roots. I need a way of showing that these are the only roots.

The factor theorem states that r is a root of a polynomial if and only if $(x-r)$ divides the polynomial. Can I use this? Yes! The only solutions to the equation would be $x=1$ and $x=4$ if this second cubic were a product of powers of $(x-1)$ and $(x-4)$. That's it.

What's the plan? Divide the cubic by one of these two linear factors. The result will be a quadratic. Factor it. Then, finally, apply the factor theorem a second time to deduce all roots.

Let's go. Ok, dividing by $x-1$ leaves the quadratic x^2-5x+5.

Check each step. I meant x^2-5x+4. This factors as $(x-1)(x-4)$.

How can I be sure? Just multiply everything out:

$$(x-1)(x-1)(x-4)=(x^2-2x+1)(x-4)=x^3-2x^2+x-4x^2+8x-4.$$

Ok, that's enough. So, I've got all the factors, which means that I have all the roots.

Can I see the proof at a glance? The roots are both visible in the sketch.

Can I prove the result differently? I didn't really think about the fact that the graph is tangent to the x-axis at 1 and crosses it at 4, but this corresponds to the fact that $x=1$ is a double root and $x=4$ a simple root.

This exercise is based on mathematics' bestselling self-help guide *How to Solve It: A New Aspect of Mathematical Method* by Georg Pólya.[177] The four step program—understand the problem, plan the solution, carry out the plan, examine the solution—has been reprinted in numerous textbooks at all levels, from kindergarten to university.

The investigation of heuristic (from "Analogy" to "Working Backwards") by Pólya, and his student Imre Lakatos, opens a window on "another face of mathematics";

> Yes, mathematics has two faces; it is the rigorous science of Euclid but it is also something else. Mathematics presented in the Euclidean way appears as a systematic, deductive science; but mathematics in the making appears as an experimental, inductive science.[178]

For a more in-depth dialectic, see 88 Dialogue or 25 Open Collaborative.

90

Retrograde

The numbers 4 and 1 are roots of a polynomial in x if the polynomial contains $-4+x$ and $-1+x$ among its factors. In particular, these numbers satisfy the equation

$$0=(-4+x)(-1+x)^2.$$

Multiplying the factors yields the polynomial equation

$$0=-4+9x-6x^2+x^3,$$

which, after adding $-2+2x$, produces the equivalent polynomial equation

$$-2+2x=-6+11x-6x^2+x^3.$$

Thus we have proved the following:

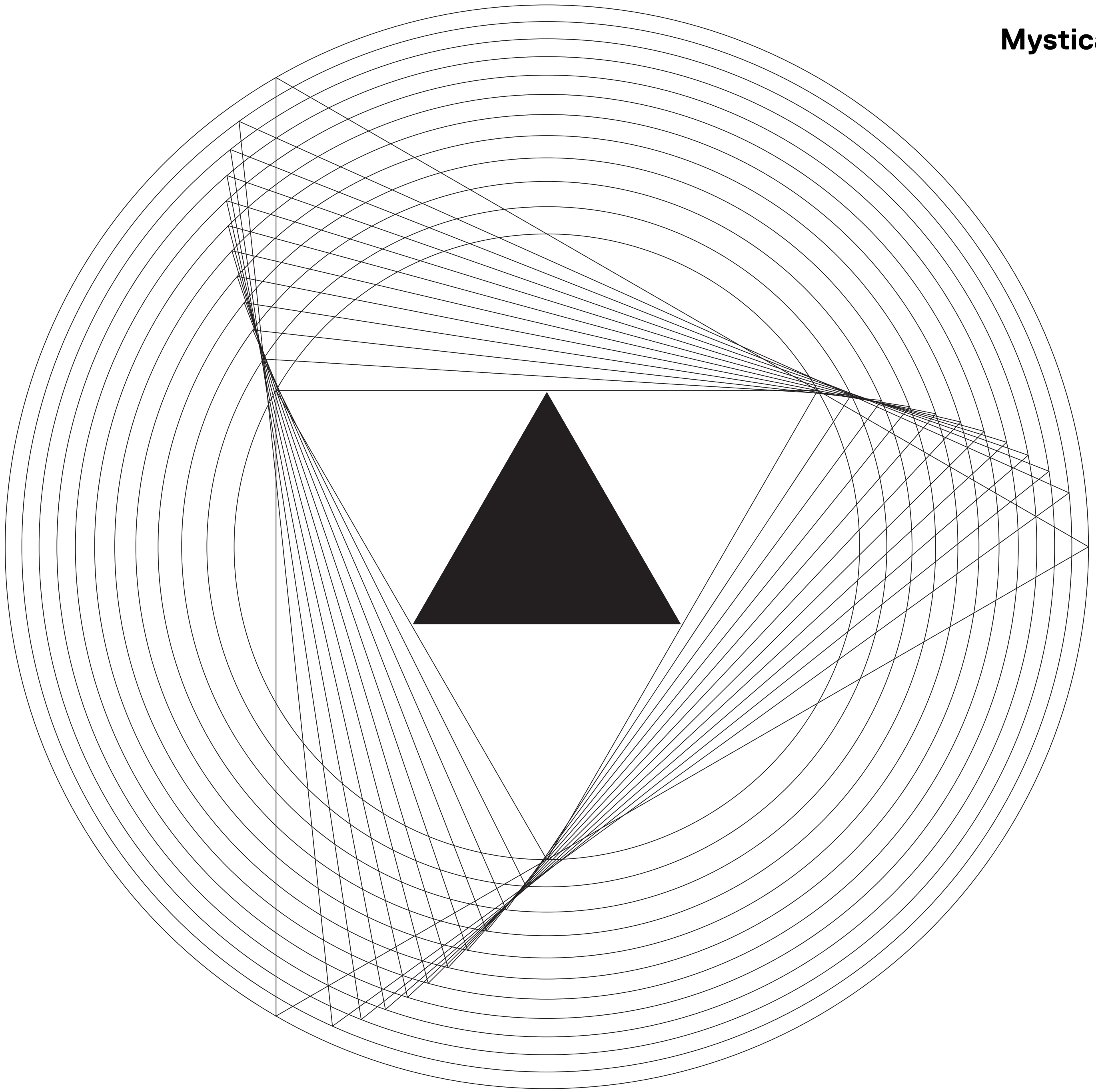
Theorem. *The unknown x is 4 or 1 when $-2+2x=-6+11x-6x^2+x^3$ for $\mathbb{R}\ni x$.*

Retrograde

It is often useful when attempting a proof to reason by steps backwards from the conclusion towards the premises. On the other hand, it is also not uncommon to find a proof that precedes the statement of a theorem as is the case here. On the level of mathematical statements and expressions, terms can also sometimes be reversed while preserving their semantic meaning. A reversal like $-4+x$ in place of $x-4$ is hardly noticed, whereas for some reason I'm never completely comfortable with the implication "Q when P" in place of "if P then Q."

91

Mystical



What does Cardano mean when he writes in his memoir that "the mystic element is necessary that we may recognize the hand of God and be taught not to forbid its workings."[179] *Necessary* for what? Salvation perhaps, but also divine inspiration. In fact, the more he describes the features of the "intuitive flash of direct knowledge," the more he finds to take personal credit for:

> The use of amplification and lucidity of understanding I have acquired partly from practice and partly at the inspiration of my spirit, for I devoted myself persistently to perfecting that intellectual flash of insight for more than forty years before I mastered it.[180]

Modern day mathematicians are more likely to recognize the flash of insight as the result of *subconscious work*.[181]

This isn't to say that modern mathematics has entirely rid itself of mystical sentiment. As a case in point, the following is an introduction to a popular mathematics book:

> In nature, we find patterns, designs and structures from the most minuscule particles, to expressions of life discernible by human eyes, to the greater cosmos. These inevitably follow geometrical archetypes, which reveal to us the nature of each form…

Actually, I'm lying. That was from an online introduction to *Sacred Geometry*.[182] (If the word "design" didn't raise an eyebrow, you probably would have figured it out had I included the closing words: "and its vibrational resonances.") But the point is that this *could* have been a popular mathematics book. It expresses a widely held faith in the unity and universality of mathematics—the language in which "that great book" is written, as Galileo famously put it.[183]

This exercise appeared by meditating on the diagram of 12 Ruler and Compass while trying to see, at a glance, the proof 67 Approximate.

Report on: *Cone of Cubics Revealed: New Insight into the Equation of Third Degree*

The subject of the paper under review is the cubic equation $x^3 - 6x^2 + 9x - 4 = 0$, and it concludes $x_1 = 1, x_2 = 1, x_3 = 4$. I have checked that the roots are correct as stated. I admit that I am somewhat mystified as to why this particular cubic is presented. If its solution is intended as an example of a general technique for solving cubics, the author should make this intention clear. Still more puzzling is the peculiar use of mathematical language and notation, which took me some time to decipher.

The author expresses the cubic above in a nonstandard form $x^3 - 6x^2 + 11x - 6x = 2x - 2$. Setting the left-hand side of this equation equal to zero, the author remarks, "The fundamental cubic defines the inverted $2/\sqrt{3}$ triangle at 2, as shown." I am unsure what is "fundamental" about $x^3 - 6x^2 + 11x - 6 = 0$ or what is being shown where. The submission does include one figure of a black triangle surrounded by numerous concentric triangles and circles, but I was unable to identify a title or caption for the image.

One could possibly interpret the "triangle" in the sense of Nickalls (1993). It is well known that a cubic with three real roots admits a solution via the cosine identity $4\cos^3\theta - 3\cos\theta - \cos 3\theta = 0$. In this case, Nickalls observed that the roots appear as the vertical projections of the vertices θ, $\theta + 2\pi/3$, $\theta + 4\pi/3$ of the equilateral triangle with center coincident with the cubic's point of inflection and circumradius equal to $2\delta := 2(b^2 - 3ac)/9a^2$. For the so-called fundamental cubic, the angle and circumradius are resp. $\theta = \pi/6$ and $2\delta = 2/\sqrt{3}$. Since the third vertex $\theta + 4\pi/3 = 3\pi/2$ of the triangle appears at the bottom of the circumscribed circle, this may account for the term "inverted."

The author identifies the roots 1, 1, and 4 of the original cubic as "a 12th part rotation and 3-fold dilation of the fundamental triangle." Taken at face value, this statement is nonsensical or, at best, false. The triangle whose vertices project to the given solutions does share the same center as the fundamental cubic (their points of inflection are vertically aligned since they have equal second derivatives) and its vertices differ by an angle $\pi/6$ (or $2\pi/12$), but their circumradii differ by a factor of $\sqrt{3}$ not 3.

In lieu of a proof or other recognizable form of justification, the paper concludes with extended remarks on the "cone of cubics." As the author does not define this object, I am unable to evaluate its merits.

Overall, I found the paper to be mostly not incorrect, though barely readable. Any genuinely new result on cubics would certainly be of interest to readers of this journal; however, I find little beyond the results already obtained in much more generality by Nickalls.

My final evaluation: I do not recommend publication.

A paper like "Cone of Cubics Revealed," which is imagined to accompany the 91 Mystical proof, wouldn't make it past a journal editor to a referee, and if it did, it's unlikely that the report would include anything beyond the final sentence. This report was modeled on the best practices advised by Chris Woodward: "A good referee report begins with a very short (a few sentences) summary of the result and the argument. It includes an opinion on whether the result and proof are (i) correct (ii) readable (iii) interesting to lots or only a few people; also (iv) a recommendation on whether it is good enough to appear in the given journal... and (v) a non-empty list of specific corrections/suggestions."[184] There doesn't seem to be consensus on what is actually required by (i), which calls into question the reliability of "the literature." See the note following 94 Authority.

For a frank discussion of one author's experience with the review process, see Robert C. Thompson's "Author vs. Referee: A Case History for Middle Level Mathematicians."[185]

93

Neologism

Definition. *Let* $k = \{\overline{x}$ *denote the* curly root *of the real number* x *if and only if* $x = k^3/(1-k)$*, where* $k < 1$*. Note that*

$$\{\overline{x}^{\,3} = x - x\,\{\overline{x}.$$

It follows that $y = -(b/a)\,\{\overline{a^3/b^2}$ *is a solution of the cubic equation* $y^3 + ay + b = 0$*.*

Theorem. *If* x *is a real number and* $x^3 - 6x^2 + 11x - 6 = 2x - 2$*, then* $x = 1$ *or* $x = 4$*.*

Proof. Substituting $x = y + 2$ reduces the given cubic equation to $y^3 - 3y - 2 = 0$. In this case, the curly root solution is $y = -(2/3)\,\{\overline{-27/4} = 2$. Dividing the cubic polynomial by $y - 2$ results in the quadratic with repeated root $y = -1$. Hence, the original cubic equation admits solutions $x = 1, 4$. □

Twentieth century mathematician and editor Ralph Boas makes the commonsense observation that inventing new words introduces something unfamiliar, which can make it harder for readers to understand: “One Bourbaki per century produces about all the neologisms that the mathematical community can absorb.”[186] (See 6 Axiomatic for more about Bourbaki.)

According to American physicist David Mermin, it requires a great deal of effort to circulate a new technical word. In “*E Pluribus Boojum*: the physicist as neologist,” he gives a detailed account of how he campaigned to get the physics community to adopt the word *boojum* for a phenomenon related to superfluidity.[187]

American Mathematician Dan Kalman coined the term *curly root* in 2009, and, last I checked, he hasn’t given up his efforts to popularize it.[188]

Of course, if $x^3 - 6x^2 + 11x - 6 = 2x - 2$, then it follows from Euler that the real number in question must be 1 or 4.

94

Authority

The eighteenth century Swiss mathematician and physicist Leonhard Euler was perhaps the most prolific mathematician of all time. One of Jean-Pierre Serre's tips from a lecture on how to write mathematics badly is that "when you give a reference, a reference that you don't want to be checked... refer to the entire works of Euler. They have not yet been published entirely."[189]

While Euler did contribute a solution to the general cubic,[190] appeals to authority don't require that the authority has actually proven the result. It may not even require that the result is true. In an opinion piece in the *Notices of the AMS*, American mathematician Melvyn Nathanson makes the following speculation:

> How do we recognize mathematical truth? If a theorem has a short complete proof, we can check it. But if the proof is deep, difficult, and already fills 100 journal pages, if no one has the time and energy to fill in the details, if a "complete" proof would be 100,000 pages long, then we rely on the judgments of the bosses in the field. In mathematics, a theorem is true, or it's not a theorem. But even in mathematics, truth can be political.[191]

When the authority of reference is oneself, a proof by authority becomes a proof by *intimidation*.[192]

95

First Person

October 23, 2015, New York

The large room is dim and sparse except for the giant CT scanner at one end. The X-ray tube and detectors begin to spin within the machine's large ring as I climb onto the narrow table that extends from its gape. Having tucked me in with a blanket and foam supports behind my knees, the technologist retreats to the control room on the other side of a lead-lined wall.

If x cubed minus six x squared plus nine x minus four is zero, then x is one or four.

There's an intercom inside the doughnut hole of the scanner, and after checking in one last time the machine takes over. "Breathe in and hold," it commands from overhead, like an assertive copilot on a very small plane. I am motionless as it calibrates to my body and the database of scans—thankfully clear—since the surgery five years ago.

The product of x minus one times x minus one times x minus four is zero.

My habit is to work with pen and paper. But I've written out the solution to this equation so many times that even lying in the scanner with arms outstretched over head I can mentally retrace each step. "Breathe." As I exhale I hardly think about tumor cells.

$$x^3-6x^2+9x-4=(x-1)(x-1)(x-4)$$

My memory relies just as much if not more on the algebraic symbols themselves—their shape in my mind's eye and their names in my ear—than on the logic of the solution. But these are meager images. What I need is a geometric proof.

The table moves me into position and begins scanning my head through a thousand sections.

A long time ago, a friend suggested an "architectural" proof that shows this factorization in one or more axonometric drawings of a cube. I try to see each term of the polynomial volumetrically, as Euclid would. I try to see the triple product in the same way. The volume degenerates at the roots. I'm lost.

The scan has progressed to my torso; my head emerges from the other end of the doughnut hole. I roll my eyes to look around. Architecturally, the scanning room is more of a low box than a cube—a cube with its top layer sectioned off.

And then, without effort, the factorization appears. Let x be greater than four. To subtract six x squared, first cut four x squared off the top, then cut one x squared from the front, and finally one x squared from the side. The low box revealed within has volume $x-4$ times $x-1$ times $x-1$.

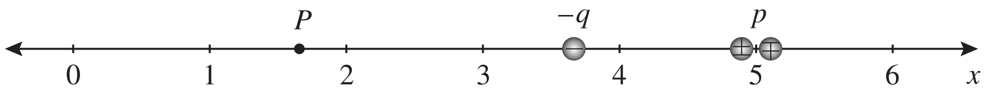
The problem is as good as solved. After a long beep, the machine declares: "Scan finished."

A more typical first-person account of mathematical discovery would follow the mythological type of the hero's journey.[193] I was having fun satirizing this style when I came across these lines from Euler's biography:

> They had 13 children altogether although only five survived their infancy. Euler claimed that he made some of his greatest mathematical discoveries while holding a baby in his arms with other children playing round his feet.[194]

How do you do mathematics after losing eight children? How do you do mathematics with five? I decided to interpret this exercise as a performative constraint instead of a literary one. If it hasn't entirely escaped the monomythic tendency, it at least complicates that stereotype of mathematical life.

For the diagram that resulted from the proof sketched here, see 62 Axonometric.



96

Electrostatic

To find the roots of the equation $x^3 - 6x^2 + 11x - 6 = 2x - 2$, place a point charge $-q$ at $x = 11/3$ and a point electric dipole p centered at $x = 5$ having two equal and opposite charges of magnitude 900 times q lying on the x-axis and separated by distance of 0.005 units. By choosing a suitable orientation for the dipole, each root of the given cubic equation will appear as a null point of the electric field on the x-axis, as measured experimentally. The electric field E_{pc} due to a point charge and the electric field E_{dp} due to the dipole are given by

$$E_{pc} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \qquad E_{dp} = \frac{1}{2\pi\epsilon_0}\frac{p}{r^3}$$

where r is the distance measured along the axis of the dipole and ϵ_0 is the permittivity of free space.

Consider the electric field at a point $P = P(x)$, where $x < 11/3$ (similar arguments apply to the remaining intervals). The field due to the point charge will be

$$\mathbf{E}_{pc}(P) = \frac{1}{4\pi\epsilon_0}\frac{q}{(x - 11/3)^2}\hat{\imath}.$$

If we align the dipole *against* the x-axis, then for $x < 11/3$ the field due to the dipole will be

$$\mathbf{E}_{dp}(P) = \frac{1}{2\pi\epsilon_0}\frac{900q \cdot 0.005}{(x-5)^3}\hat{\imath}.$$

According to the principle of superposition, $\mathbf{E} = \mathbf{E}_{pc} + \mathbf{E}_{dp}$. Hence the electric field is zero at $P(x)$ with $x < 11/3$ if x satisfies

$$\begin{aligned}
\mathbf{E}_{pc}(P) &= -\mathbf{E}_{dp}(P)\\
\frac{1}{4\pi\epsilon_0}\frac{q}{(x-11/3)^2}\hat{\imath} &= -\frac{1}{2\pi\epsilon_0}\frac{900q \cdot 0.005}{(x-5)^3}\hat{\imath}\\
\frac{1}{(x-11/3)^2} &= \frac{-9}{(x-5)^3}\\
(x-5)^3 &= -9(x-11/3)^2\\
x^3 - 15x^2 + 75x - 125 &= -9x^2 + 66x - 121\\
x^3 - 6x^2 + 11x - 6 &= 2x - 2
\end{aligned}$$

and thus x is a root of the given equation, as claimed.

A physics colleague suggested to me the idea of using electrostatics to represent a cubic equation. A similar demonstration shows that null points satisfy the cubic equation when $x > 5$. For the interval $11/3 < x < 5$, we must align the dipole *with* the x-axis. Since neither $x = 11/3$ nor $x = 5$ satisfies the given equation, all roots will be found in this manner. Derivations of the electric fields of a point charge and a point electric dipole can be found in most college physics textbooks.[195]

This proof reminds me of the following passage in Henri Poincaré's essay "Intuition and Logic in Mathematics," where he presents his first example of the intuitive tendency:

> [L]ook at Professor [Felix] Klein: he is studying one of the most abstract questions of the theory of functions to determine whether on a given Riemann surface there always exists a function admitting of given singularities. What does the celebrated German geometer do? He replaces his Riemann surface by a metallic surface whose electric conductivity varies according to certain laws. He connects two of its points with the two poles of a battery. The current, says he, must pass, and the distribution of this current on the surface will define a function whose singularities will be precisely those called for by the enunciation.
>
> Doubtless Professor Klein well knows he has given here only a sketch: nevertheless he has not hesitated to publish it; and he would probably believe he finds in it, if not a rigorous demonstration, at least a kind of moral certainty. A logician would have rejected with horror such a conception, or rather he would not have had to reject it, because in his mind it would never have originated.[196]

97

Psychedelic

The image was created from Newton's method for finding roots. Newton's method applies a function to an initial estimate of a root and returns another, hopefully improved, root approximation. By applying the function to this second estimate, and so on, one obtains a sequence of approximations that often converge to an actual root, as was shown in 76 Experimental. The image in this proof was obtained by taking points of the *complex* plane as initial estimates. After some number of iterations, the method agrees with one or the other root to within four decimal places. The wavey "C" curve on the right side of the image separates initial points converging to 1 (left) from those points converging to 4 (right). If the number of iterations needed to arrive (sufficiently close) to a root is even/odd then the point is colored black/white.

The inspiration for this style came from an article entitled "Mathematics and the Psychedelic Revolution" by Ralph Abraham of the Department of Mathematics at UC Santa Cruz.[197] The fascinating account appeared in the *Bulletin of the Multidisciplinary Association for Psychedelic Studies*:

> It all began in 1967 when I was a professor of mathematics at Princeton, and one of my students turned me on to LSD. That led to my moving to California a year later, and meeting at UC Santa Cruz a chemistry graduate student who was doing his Ph.D. thesis on the synthesis of DMT. He and I smoked up a large bottle of DMT in 1969, and that resulted in a kind of secret resolve, which swerved my career toward a search for the connections between mathematics and the experience of the logos, or what Terence calls "the transcendent other." This is a hyperdimensional space full of meaning and wisdom and beauty, which feels more real than ordinary reality, and to which we have returned many times over the years, for instruction and pleasure. In the course of the next twenty years there were various steps I took to explore the connection between mathematics and the logos.... There is no doubt that the psychedelic revolution in the 1960s had a profound effect on the history of computers and computer graphics, and of mathematics, especially the birth of postmodern maths such as chaos theory and fractal geometry. This I witnessed personally. The effect on my own history, viewed now in four decades of retrospect, was a catastrophic shift from abstract pure math to a more experimental and applied study of vibrations and forms, which continues to this day.

The use of prescription-strength stimulants among mathematicians is probably somewhat more common (though how common would be difficult to know) than the use of hallucinogenics.[198] J. E. Littlewood offered a more cautiously optimistic note on academic drug use: "I can envisage a future in which stimulant drugs could raise mental activity for a set period of work, and relaxing ones give a suitable compensating period, perhaps of actual sleep. The present is a time of transition; stimulants do exist; but they should be used only with the greatest care and only in a crisis. And there is the problem of knowing what *is* a crisis."[199]

98

Mondegreen

Their omelette: eggs, beer, eel.
If eggs cued my nose, six eggs queered.
Plus nine eggs.
My nose for equals—zero.
Then, egg sequels one or four.
Proofs pose eggs irrational.
Irrational root of a manic pollen oatmeal is a divine, sore Ovid, sconced in turn.
For their moor, the all-terraining koi fish aunts imply a positive route.
Sin stew is not a solution, another route must be ear-rational.
Butter rational routes.
Come in, conjugal pairs!

Mondegreen

A mondegreen is an accidental homophonic translation. This variation is written in the spirit of irreverence towards language that Lewis Carroll and his Humpty Dumpty share with so many mathematicians:

> "When *I* use a word," Humpty Dumpty said, in rather a scornful tone, "it means just what I choose it to mean—neither more nor less."
>
> "The question is," said Alice, "whether you *can* make words mean so many different things."
>
> "The question is," said Humpty Dumpty, "which is to be master—that's all."[200]

As for Carroll, he was no less explicit in his views on the matter. In *Symbolic Logic*, he writes:

> The writers, and editors, of the Logical textbooks which run in the ordinary grooves—to whom I shall hereafter refer by the (I hope inoffensive) title "The Logicians"—. . . speak of the Copula of a Proposition "with bated breath"; almost as if it were a living, conscious Entity, capable of declaring for itself what it chose to mean, and that we, poor human creatures, had nothing but to ascertain *what* was its sovereign will and pleasure, and submit to it.
>
> In opposition to this view, I maintain that any writer of a book is fully authorised in attaching any meaning he likes to any word or phrase he intends to use.[201]

I had hoped for this homophonic translation (46 Cute is the source) to illustrate the fully authorized author in action—how language can be made to bend to our will—but I fear that the experiment may have backfired. Whenever I see the ubiquitous rubric "**Theorem**. *Let*. . . ," I cannot help but hear "their omelette." Sometimes I even smell it cooking, all buttery in the pan.

99

Prescribed

Theorem. *Let x be a real number. If $x^3 - 6x^2 + 11x - 6 = 2x - 2$, then $x = 1$ or $x = 4$.*

Proof. The proof is left to the reader. $\square$

Remember, don't throw away
The quadrant of unused situations just because they're here:
They may not always be, and you haven't finished looking
Through them all yet. So much that happens happens in small ways
That someone was going to get around to tabulate, and then never did,
Yet it all bespeaks freshness, clarity and an even motor drive
To coax us out of sleep and start us wondering what the new round
Of impressions and salutations is going to leave in its wake
This time. And the form, the precepts, are yours to dispose of as you will,
As the ocean makes grasses, and in doing so refurbishes a lighthouse
On a distant hill, or else lets the whole picture slip into foam.

—John Ashbery, "Someone You Have Seen Before" (excerpt)[202]

Postscript

Where did the equation $x^3 - 6x^2 + 11x - 6 = 2x - 2$ come from? Its particulars were chosen based on an algebro-geometric reading of the story that forms the basis of Raymond Queneau's *Exercises in Style*. The two points of intersections or solutions encode the two sightings of the man in the felt hat. The first intersection accounts for the first half of the story, when the long-neck is traveling in the same direction as the narrator, and hence the curves are tangent. The second, transverse intersection marks the second half of the story, when the narrator passes the man standing in front of the train station. The two sides of the equation emerged as the simplest curves that conformed to these conditions. Only in rereading Queneau's book after having drafted these exercises, I realize that the subject of his narrative, "the chap in question," is the picture of bad taste. He's a "giraffe" in a funny hat and a coat with a button in the wrong place and has such poor manners that he gets in arguments on buses. And as it happens, this degenerate cubic equation, with its awkward, non-standard form, similarly accentuates stylistic variation.

Acknowledgments

One of the most satisfying aspects of writing this book has been the opportunity to meet and discuss mathematics with people with whom I never otherwise would have done that. It is thanks to the conversations and support of many acquaintances, friends, and family that it became a reality.

In the very earliest stage of the project, through the support of the City University of New York, I was able to participate in a mathematics writing group led by Roman Kossak, whose guidance, friendship, and example have been critical to the project ever since. I am especially grateful to Piet Hut and the Program in Interdisciplinary Studies at The Institute for Advanced Study, Princeton for hosting me during the Spring of 2012 and giving the project real momentum. There, I met Monica Manolescu and Siobhan Roberts, who made several helpful comments on a draft of the book proposal, and I am indebted to Siobhan for introducing me to my editor Vickie Kearn at Princeton University Press. I received many helpful comments and suggestions on a partial draft of the exercises at a 2016 Workshop on Creative Writing in Mathematics and Mathematical Sciences, which was organized by Marjorie Senechal at the Banff International Research Station. My home institution, Sarah Lawrence College, has generously supported this research, and I am thankful for helpful conversations with colleagues Dan King, Michael Siff, Jim Marshall, Melissa Frazier, Scott Calvin, Jason Earle, Melvin Bukiet, and Angela Ferraiolo. Without the tireless research assistance of several students past and present this would have been a lesser book; I owe my sincerest gratitude to Ella Pavlechko, Xueyi Bu, Lauren Bréard, Isak McCune, Sarah Dennis, and Marshall Pangilinan. Avinoam Hennig and Emily Rogers also made helpful suggestions.

Thanks go to Sonja Mapes for putting me in touch with James Nickerson and Regina Nuzzo at Gallaudet University about mathematics in ASL. I am especially grateful to Christopher Hayes who gave up time studying for graduate school qualifying exams to translate a proof into ASL. It is a pleasure to thank Peggy Swartzel Lott for the beautiful sign language illustrations that she, along with Daniel W. Renner and Rob Hills made. Bart Van Steirteghem and my cousin Anne-Laure Tapernoux helped me with the French translation, and Eric Wambach and Alexander Jermann with the German. I'm grateful to Roman Kossak and, separately, John P. Colvis for letting me reproduce their correspondence. Dan Kalman kindly shared his typesetting macros for curly-roots. My brother David had thoughtful suggestions when I was stuck on the problem of ordering the variations, and my oldest brother Emmanuel didn't balk when one August weekend while visiting his home in Massachusetts I asked him to lend his professional expertise and camera equipment for the afternoon to make the photographs for 66 Hand-Waving. I am thankful to both, and to our parents Jon and Marie-Claire for their encouragement. My friends Bart Van Steirteghem, Spencer Gerhardt, Brian O'Connell, and

Helena Kauppila have been consistent sources of support and feedback on drafts. I also received insightful comments on the manuscript from Dylan Baker-Rice, Peter Macapia, David Reinfurt, Tom La Farge, Wendy Walker, Daniel Levin Becker, Marjorie Senechal, Siobhan Roberts, Étienne Ghys, Craig R. Whitney, and two anonymous reviewers.

The entire staff at PUP labored to make the book more beautiful than I could have imagined, and I am deeply thankful to the designer Chris Ferrante for his attention to all the details of both content and form. Lor Campbell Gehret expertly copyedited the manuscript. I am more than grateful to Vickie Kearn not only for agreeing to publish this work on the basis of a slim proposal but also for her enthusiasm and continued support, which made me feel I was writing something that no one else could.

None of this would have come together without the love, attention, patience, and impeccable standards of my innermost circle—Alexandra Whitney and our sons Max and Roland.

Notes

1. Galileo Galilei, *Sidereus Nuncius, or The Sidereal Messenger*, p. 98. Quoted in Sobel, *Galileo's Daughter: A Historical Memoir of Science, Faith and Love*.
2. Here I am paraphrasing the twentieth century mathematician and philosopher Gian-Carlo Rota's *Indiscrete Thoughts*; see the comment following 6 Axiomatic for the quote.
3. Le Lionnais, "Lipo: First Manifesto."
4. Here I am paraphrasing the preface of Philip Davis and Reuben Hersh's wonderful 1981 book *The Mathematical Experience*, which has served me as both a style guide to writing about mathematics and a source of encouragement throughout this project.
5. Pétard, "A Contribution to the Mathematical Theory of Big Game Hunting."
6. Calvino, *Six Memos for the Next Millennium*, p. 43.
7. Ibid, p. 43.
8. Ibid, p. 43.
9. Michael Harris has noted, "Aesthetic judgement in mathematics is hampered by its meager lexicon; it doesn't inspire 'lofty' habits in the use of language" and "we have made no progress at all toward elucidating what beauty in mathematics has to do with beauty elsewhere." Harris, *Mathematics without Apologies: Portrait of a Problematic Vocation*, p. 307.
10. The American mathematician William Thurston discusses the importance of "tone and flavor" in his 1994 article "On Proof and Progress in Mathematics." See the note following 33 Calculus for the full quote.
11. Michael Artin, *Algebra*, p. 352. For other sorts of omission see 44 Omitted with Condescension and 99 Prescribed.
12. Grosholz, *Representation and Productive Ambiguity in Mathematics and the Sciences*. In the preface, she writes that "reductive methods are successful at problem-solving not because they eliminate modes of representation, but because they multiply and juxtapose them; and this often creates what I call productive ambiguity."
13. Tony Padilla discusses the paper Lander and Parkin, "Counterexample to Euler's conjecture on sums of like powers" in an episode on YouTube *Numberphile*: Haran, *The Shortest Papers Ever: Numberphile*.
14. Herbst, "Establishing a Custom of Proving in American School Geometry: Evolution of the Two-Column Proof in the Early Twentieth Century," p. 287.
15. Ghys, "Inner Simplicity vs. Outer Simplicity," p. 7.
16. Pólya, *How to Solve It: A New Aspect of Mathematical Method*, pp. 172–173.
17. Dudley, "What is Mathematics For?", p. 613.
18. Ewald, *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*, p. 1108.
19. Peano and Kennedy, *Selected Works of Giuseppe Peano*, p. 101.
20. For comparison, Alfred North Whitehead and Bertrand Russell define the number 1 with proposition Z52.16 on page 345 of their *Principia Mathematica*, Whitehead and Russell, *Principia Mathematica to *56*.
21. The Bourbaki group began in 1934 and included Henri Cartan, Claude Chevalley, Jean Delsarte, Jean Dieudonnè, and Andrè Weil among others. See Senechal, "Mathematical Communities: The Continuing Silence of Bourbaki—An Interview with Pierre Cartier, June 18, 1997."
22. Bourbaki, "The Architecture of Mathematics," p. 231.
23. Rota, *Indiscrete Thoughts*, p. 142.

24. Cardano, *Hieronymi Cardani, Praestantissimi Mathematici, Philosophi, Ac Medici, Artis Magnae Sive De Regvlis Algebraicis Lib. unus: qui & totius operis de arithmetica, quod opus perfectum inscribitur, est in ordine decimus*, p. 41. The image here is courtesy of the Rare Book & Manuscript Library, Columbia University.
25. Cardano, *Ars Magna or the Rules of Algebra*, p. 139.
26. Ibid., p. 8.
27. These references, Artin, *Algebra*; Herstein, *Topics in Algebra* are listed on the Columbia University Mathematics graduate program's webpage for prospective students, "What PhD Graduates Are Assumed to Know."
28. Conway and Shipman, "Extreme Proofs I: The Irrationality of $\sqrt{2}$," p. 2.
29. Decker, *Swatting flies with a sledgehammer*.
30. Roberts, *Genius at Play: The Curious Mind of John Horton Conway*, p. 286.
31. Queneau, "Letters, Numbers, Forms: Essays 1928–1970," p. 99. This paper appears alongside the Bourbaki manifesto, "The Architecture of Mathematics," (see the discussion following 6 Axiomatic) in *Great Currents of Mathematical Thought*, edited by François Le Lionnais.
32. Gardner, "Mathematical Games: 'Look-see' proofs that offer visual proof of complex algebraic formulas"; Isaacs, "Two Mathematical Papers Without Words"; Nelson, *The Penguin Dictionary of Mathematics*.
33. Doyle et al., *Proofs Without Words and Beyond*.
34. New York State Education Department, *Past Examinations*.
35. Gleason, "Angle Trisection, the Heptagon, and the Triskaidecagon."
36. Pólya, *How to Solve It: A New Aspect of Mathematical Method*, p. 162.
37. Bobzien, "Ancient Logic." See also Lodder, *Deduction through the ages: A history of truth*.
38. The details of the method of computing eigenvalues can be found in any basic linear algebra textbook. Linear algebra was part of my undergraduate physics coursework, and I've based this proof on the textbook that my mathematical methods professor Mike Hobson co-authored. Riley, Hobson, and Bence, *Mathematical Methods for Physics and Engineering: A Comprehensive Guide*, pp. 220–222. The idea for this proof was suggested to me by Joseph Maher.
39. Rabouin, "Styles in Mathematical Practice," p. 233.
40. Mazur, "*History of Mathematics* as a Tool," p. 2.
41. See Høyrup, "The Babylonian Cellar Text BM 85200 + VAT 6599 Retranslation and Analysis" and O'Connor and Robertson, *The MacTutor History of Mathematics Archive: An Overview of Babylonian Mathematics*.
42. Høyrup, "The Babylonian Cellar Text BM 85200 + VAT 6599 Retranslation and Analysis."
43. Boute, "How to Calculate Proofs: Bridging the Cultural Divide."
44. Lamport, "How to Write a Proof," p. 300.
45. See, for example, Serre, *How to Write Mathematics Badly*.
46. The Oulipo called this exercise "larding" or "PALF" (*production automatique de la langue Française*). Becker, *Many Subtle Channels: In Praise of Potential Literature*, p. 175. My sources for definitions are Wikipedia and Nelson, *The Penguin Dictionary of Mathematics*.
47. Rota, *Indiscrete Thoughts*, p. 146.
48. Barany and MacKenzie, "A Dusty Discipline," pp. 3–4. See also Barany and MacKenzie, "Chalk: Materials and Concepts in Mathematics Research," p. 122: "Mathematical work

rests on self-effacing technologies of representation that seem to succeed in removing themselves entirely from the picture at the decisive junctures of mathematical understanding. It is only by virtue of these disappearing media that one can be said to understand a concept itself rather than its particular manifestations."

49. Jackson, "Teaching Math in America: An Exhibit at the Smithsonian."
50. Goldman, "Inside a Mathematical Proof Lies Literature, Says Stanford's Reviel Netz."
51. Goethe, *Maxims and Reflections*, #1279.
52. van der Waerden, *Algebra*, §5.7.
53. Stedall, *From Cardano's Great Art to Lagrange's Reflections: Filling a Gap in the History of Algebra*. For a partial translation of Lagrange and Serret, *Oeuvres de Lagrange*, see Ng, *Lagrange's Work on General Solution Formulae for Polynomial Equations*.
54. Gowers, "Is Massively Collaborative Mathematics Possible?"
55. Editorial, *Nature*, "Parallel Lines," p. 408.
56. Ball, "Strength in Numbers."
57. Gowers and Nielsen, "Massively Collaborative Mathematics," p. 880.
58. Lakatos, Worrall, and Zahar, *Proofs and Refutations: The Logic of Mathematical Discovery*.
59. Cohen, *Computer Algebra and Symbolic Computation: Mathematical Methods*.
60. Gilbreth and Gilbreth, "Process Charts—First Steps in Finding the One Best Way," p. 3.
61. Sleeman, "Solving Linear Algebraic Equations"; Lacey, *Flowcharting Proofs*.
62. Emch, "New Models for the Solution of Quadratic and Cubic Equations."
63. Sharp, *Surfaces: Explorations with Sliceforms*.
64. See also Cundy and Rollett, *Mathematical Models*, p. 197.
65. Cardano, *Ars Magna or the Rules of Algebra*, pp. 217–219.
66. Lakatos, Worrall, and Zahar, *Proofs and Refutations: The Logic of Mathematical Discovery*.
67. Thurston, "On Proof and Progress in Mathematics," pp. 163–164.
68. Ibid, pp. 163–167.
69. Fibonacci, *Fibonacci's Liber Abaci: A Translation Into Modern English of Leonardo Pisano's Book of Calculation*.
70. Grant, *A Source Book in Medieval Science*, p. 243.
71. Brown and Brunson, "Fibonacci's Forgotten Number."
72. Shen et al., *The Nine Chapters on the Mathematical art: Companion and Commentary*.
73. Ellisllk, *Oumathpo*.
74. Duchêne and Leblanc, *Rationnel mon Q*.
75. You, "Who Are the Science Stars of Twitter?"
76. Matthews and Brotchie, *Oulipo Compendium*, p. 326.
77. Oulipo, *Un Certain Disparate: Entretiens avec François Le Lionnais*.
78. archive, *Mathematics: Article Statistics for 2016*.
79. Chang, "Marina Ratner, Émigré Mathematician Who Found Midlife Acclaim, Dies at 78," *The New York Times* Science section, July 25, 2017.
80. For an axiomatic approach to origami and its relationship to classical constructions, see Alperin, "A Mathematical Theory of Origami Constructions and Numbers."
81. Hull, "Solving Cubics with Creases: The Work of Beloch and Lill."
82. Borovik, *Mathematics Under the Microscope: Notes on Cognitive Aspects of Mathematical Practice*, pp. xiv–xv.
83. Cajori, "Origin of the Name 'Mathematical Induction'," pp. 198–199.

84. Kolata, "At Last, Shout of 'Eureka!' in Age-Old Math Mystery"; Robinson, "Russian Reports He Has Solved a Celebrated Math Problem"; Chang, "A Possible Breakthrough in Explaining a Mathematical Riddle."
85. Robbins, *This Is a News Website Article About a Scientific Paper*.
86. Nordgaard, "Sidelights on the Cardan-Tartaglia Controversy."
87. Chevalley, "Variations du style mathématique" as translated in Mancosu, *Mathematical style*.
88. Motion Pictures Arts and Sciences, *Screenwriting Resources*.
89. Descartes, *The Geometry of Rene Descartes*, p. 10.
90. Khayyam, *Algebra wa Al-Muqabala: An Essay by the Uniquely Wise 'Abel Fath Omar Bin Al-Khayyam on Algebra and Equations*, p. 41.
91. Doyle, *The Memoirs of Sherlock Holmes*, p. 40.
92. Stillwell, *Mathematics and Its History (Undergraduate Texts in Mathematics)*, p. 56.
93. O'Connor and Robertson, The MacTutor History of Mathematics Archive: François Viète.
94. See Arana, "On the Alleged Simplicity of Impure Proof."
95. Wiedijk, *The Seventeen Provers of the World*.
96. Krantz, *The Proof is in the Pudding: A Look at the Changing Nature of Mathematical Proof*, pp. 199–201.
97. Zeilberger, *Dr. Z's Opinions*.
98. Apart from this note, the proof and comment reproduce my correspondence with Mr. John P. Colvis verbatim. *The New York Times* article is Thompson, "The Year in Ideas: Outsider Math," and an account of Perko appears in Johnson, *Debunking knot theory's favourite urban legend*.
99. Byrne, *The First Six Books of the Elements of Euclid: In Which Coloured Diagrams and Symbols Are Used Instead of Letters for the Greater Ease of Learners*, p. vii.
100. Feynman, *What Do You Care What Other People Think?*, p. 59.
101. Fine and Rosenberger, *The Fundamental Theorem of Algebra*, pp. 134–136.
102. Pétard, "A Contribution to the Mathematical Theory of Big Game Hunting," p. 447.
103. Heath, *The Thirteen Books of Euclid's Elements*.
104. Lakatos, Worrall, and Zahar, *Proofs and Refutations: The Logic of Mathematical Discovery*, p. 142.
105. Clay Mathematics Institute, *Euclid's Elements*.
106. Heath, *The Thirteen Books of Euclid's Elements*, p. 129.
107. Gentzen, "Untersuchungen über das logische Schließen (Investigations into Logical Inference)."
108. McFarland, "The Unstoppable TI-84 Plus: How an Outdated Calculator Still Holds a Monopoly on Classrooms."
109. Wikipedia, *Texas Instruments signing key controversy*.
110. Pólya, *How to Solve It: A New Aspect of Mathematical Method*, p. 121.
111. U.S. Patent and Trademark Office, *United States Patent Classification Class Numbers and Titles*.
112. Walster and Hansen, *Solving a Nonlinear Equation Through Intervalic Arithmetic and Term Consistency*.
113. Lill, "Résolution graphique des équations numériques de tous les degrés à une seule inconnue, et description d'un instrument inventé dans ce but."
114. van der Waerden et al., *Moderne Algebra*.

115. Mac Lane, "Van der Waerden's *Modern Algebra*," pp. 321–322.
116. Most abstract algebra textbooks, for example, Artin, *Algebra*, include an introduction to the basic notions of algebraic geometry.
117. Eisenman, "From Object to Relationship II: Casa Giuliani Frigerio: Giuseppe Terragni Casa Del Fascio," p. 41.
118. Morrison, "Fermi Questions."
119. Slupinski and Stanton, "The Special Symplectic Structure of Binary Cubics."
120. Thurston, "On Proof and Progress in Mathematics," p. 166.
121. Barany, "Mathematical Research in Context."
122. Rota, *Indiscrete Thoughts*, p. 8.
123. Lakoff and Núñez, *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being*, pp. 37–39, italics in original.
124. Wilkinson, "The Perfidious Polynomial," pp. 2–3.
125. Sangwin, "Modelling the Journey from Elementary Word Problems to Mathematical Research," p. 1444.
126. Devlin, *The Problem with Word Problems*.
127. Ifrah, *The Universal History of Numbers*, p. 431.
128. Samuelson, "How Deviant Can You Be?"
129. "I prefer the designation 'Islamic' for that civilization who's mathematical achievements I shall describe. For, although it was home to men and women of many different races and faiths, its essential features were defined by those who professed the Islamic belief that 'There is no god but God and Muhammad is the Messenger of God'." Berggren, *Episodes in the Mathematics of Medieval Islam*, p. viii.
130. I worked primarily from Daoud Kasir's translation Khayyam, *The Algebra of Omar Khayyam*, p. 99 and also consulted Khayyam, *Algebra wa Al-Muqabala: An Essay by the Uniquely Wise 'Abel Fath Omar Bin Al-Khayyam on Algebra and Equations* and Henderson, *Geometric Solutions of Quadratic and Cubic Equations*.
131. Heath, *Apollonius of Perga: Treatise on Conic Sections with Introductions Including an Essay on Earlier History on the Subject*, p. 59.
132. Berggren, *Episodes in the Mathematics of Medieval Islam*, p. 124.
133. See p. 2 of Dieudonné, *History of Algebraic Geometry* for the quote about Apollonius.
134. Rashed, *The Development of Arab Mathematics: Between Arithmetic and Algebra*, pp. 332–349.
135. Ibid., p. 338.
136. Audin, *Conseils aux auteurs de textes mathématiques*.
137. van der Waerden et al., *Moderne Algebra*.
138. Tardy, "The Role of English in Scientific Communication: Lingua Franca or Tyrannosaurus Rex?", p. 249.
139. Bieberbach, "Stilarten mathematischen Schaffens" quoted in Mancosu, *Mathematical Style*.
140. Glaser, "A Study of Perceptions of Mathematical Signs: Implications for Teaching."
141. Cavendar, A., D.S. Otero, J.P. Bigham, R.E. Ladner, "ASL-STEM Forum: Enabling Sign Language to Grow through Online Collaboration," p. 2075.
142. Glaser, "A Study of Perceptions of Mathematical Signs: Implications for Teaching."
143. Nickerson, Private Correspondence.
144. For details, see Higgins, "Slide-Rule Solutions of Quadratic and Cubic Equations."
145. National Air and Space Museum, *Slide Rule, 5-Inch, Pickett N600-ES, Apollo 13*.

146. Halmos, *I Want to be a Mathematician: An Automathography*, p. 321.
147. Papakonstantinou and Tapia, "Origin and Evolution of the Secant Method in One Dimension."
148. Cajori, "Historical Note on the Newton-Raphson Method of Approximation."
149. Hurd, "A Note on Early Monte Carlo Computations and Scientific Meetings."
150. Metropolis and Ulam, "The Monte Carlo Method," p. 339.
151. Smith, "Editor's Note," reproduced in Jensen, "The Laguerre-Samuelson Inequality with Extensions and Applications in Statistics and Matrix Theory," p. 20.
152. Regazzini, "Probability and Statistics in Italy During the First World War I. Cantelli and the Laws of Large Numbers" cited by O'Connor and Robertson, *The MacTutor History of Mathematics Archive: Francesco Paolo Cantelli*.
153. Brouwer and Dalen, *Brouwer's Cambridge Lectures on Intuitionism*.
154. Heyting, *Intuitionism: An Introduction*, p. 8.
155. See Iemhoff, "Intuitionism in the Philosophy of Mathematics."
156. Sobel, *Galileo's Daughter: A Historical Memoir of Science, Faith and Love*, p. 39 and Brown, *Galileo's Anagrams and the Moons of Mars*.
157. Huygens, "De Saturni Luna Observatio Nova."
158. Fauvel and Gray, *The History of Mathematics: A Reader*, p. 407.
159. Archimedes, *The Works of Archimedes: Volume 2, On Spirals*, p. 28.
160. Grattan-Guinness, *The Norton History of the Mathematical Sciences: The Rainbow of Mathematics*, p. 106.
161. Tartaglia, *Quesiti et inventioni diverse*, p. 120.
162. Gutman, "Quando Che'l Cubo."
163. Fadiman, *Fantasia Mathematica*, p. 268.
164. Beziau, Chakraborty, and Dutta, *New Directions in Paraconsistent Logic*, p. 64.
165. Villani, *Birth of a Theorem: A Mathematical Adventure*, p. 251.
166. Campbell-Kelly, *The History of Mathematical Tables: From Sumer to Spreadsheets*, p. 27.
167. Nogrady, "A New Method for the Solution of Cubic Equations."
168. Salzer, Richards, and Arsham, *Table for the Solution of Cubic Equations*.
169. Lamb, "Two-Hundred-Terabyte Maths Proof is Largest Ever."
170. Grattan-Guinness, *The Search for Mathematical Roots, 1870–1940: Logics, Set Theories and the Foundations of Mathematics from Cantor through Russell to Godel*, p. 208 is quoted by mixedmath, *Mathematics StackExchange*.
171. Queneau, *Les Fondements de la littérature: d'aprés David Hilbert*; Hilbert, Bernays, and Unger, *The Foundations of Geometry*.
172. Heath, *The* Method *of Archimedes*.
173. Gowers, *Discovering a Formula for the Cubic*.
174. Cullen, *Astronomy and Mathematics in Ancient China: The 'Zhou Bi Suan Jing'*, p. 174, tr. Cullen.
175. Linderholm, *Mathematics Made Difficult*, p. 10.
176. Zukav, *The Dancing Wu Li Masters*, p. 208.
177. Pólya, *How to Solve It: A New Aspect of Mathematical Method*.
178. Ibid., p. vii.
179. Cardano, *Ars Magna or the Rules of Algebra*, p. 243.
180. Ibid., pp. 246–7.

181. For example, see Littlewood and Bollobás, *Littlewood's Miscellany*, p. 196 and Pólya, *How to Solve It: A New Aspect of Mathematical Method*, pp. 197–9.
182. Rawles, *Sacred Geometry Introductory Tutorial*.
183. Galileo Galilei, *Discoveries and Opinions of Galileo*, p. 238.
184. Woodward, *Mathoverflow*.
185. Thompson, "Author vs. Referee: A Case History for Middle Level Mathematicians."
186. Boas, "Can We Make Mathematics Intelligible?", p. 728.
187. Mermin, "E Pluribus Boojum: The Physicist as Neologist."
188. Kalman, *Uncommon Mathematical Excursions: Polynomia and Related Realms*; Kalman and Burke, "Solving Cubic Equations with Curly Roots."
189. Serre, *How to Write Mathematics Badly*.
190. Kalman, *Uncommon Mathematical Excursions: Polynomia and Related Realms*, p. 78.
191. Nathanson, "Desperately Seeking Mathematical Truth," p. 773. "Many (I think most) papers in most refereed journals are not refereed. There is a presumptive referee who looks at the paper, reads the introduction and the statements of the results, glances at the proofs, and, if everything seems okay, recommends publication. Some referees do check proofs line-by-line, but many do not. When I read a journal article, I often find mistakes. Whether I can fix them is irrelevant. The literature is unreliable."
192. Gian-Carlo Rota's graduate student recollections include the following origin story for this phenomenon: "[William] Feller took umbrage when someone interrupted his lecture by pointing out some glaring mistake. He became red in the face and raised his voice, often to full shouting range. It was reported that on occasion he had asked the objector to leave the classroom. The expression "proof by intimidation" was coined after Feller's lectures (by Mark Kac). During a Feller lecture, the hearer was made to feel privy to some wondrous secret, one that often vanished by magic as he walked out of the classroom at the end of the period. Like many great teachers, Feller was a bit of a con man." Rota, *Indiscrete Thoughts*, pp. 8–9.
193. For example, Villani, *Birth of a Theorem: A Mathematical Adventure*.
194. O'Connor and Robertson, *The MacTutor History of Mathematics Archive: Leonhard Euler*.
195. For example Halliday, Walker, and Resnick, *Fundamentals of Physics Extended*.
196. Poincaré, *The Value of Science: Essential Writings of Henri Poincaré*, p. 198.
197. Abraham, "Mathematics and the Psychedelic Revolution: Recollections of the Impact of the Psychedelic Revolution on the History of Mathematics and My Personal Story."
198. Amphetamines were a part of Paul Erdős' regimen: "10 to 20 milligrams of Benzedrine or Ritalin, strong espresso, and caffeine tablets." Hoffman, *The Man Who Loved Only Numbers: The Story of Paul Erdős and the Search for Mathematical Truth*, p. 7.
199. Littlewood and Bollobás, *Littlewood's Miscellany*, p. 200.
200. Carroll, *Alice in Wonderland (Norton Critical Editions)*, p. 163.
201. Quoted in Carroll and Gardner, *The Annotated Alice: The Definitive Edition*, p. 214.
202. Ashbery, *Notes from the Air*, p. 16.

Sources

Abraham, Ralph. "Mathematics and the Psychedelic Revolution: Recollections of the Impact of the Psychedelic Revolution on the History of Mathematics and My Personal Story." *MAPS Bull.* 18, no. 1 (2008) 8–10.

Alperin, Roger C. "A Mathematical Theory of Origami Constructions and Numbers." *New York Journal of Mathematics* 6 (2000): 119–133.

Arana, Andrew. "On the Alleged Simplicity of Impure Proof." In *Simplicity: Ideals of Practice in Mathematics and the Arts*, edited by Roman Kossak and Philip Ording, 205–226. New York: Springer, 2017.

Archimedes. *The Works of Archimedes: Volume 2, On Spirals*. Translated with commentary by Reviel Netz. Cambridge, UK: Cambridge University Press, 2017.

archive, arXiv.org e-Print. *Mathematics: Article Statistics for 2016.* https://arxiv.org/year/math/16, 2017. Accessed July 28, 2017.

Artin, Michael. *Algebra*. New York: Prentice Hall, 1991.

Ashbery, John. *Notes from the Air*. New York: HarperCollins, 2007.

Audin, Michèle. Conseils aux auteurs de textes mathématiques, October 1997. irma.math.unistra.fr/~maudin/newhowto.ps. Accessed June 14, 2018.

Ball, Philip. "Strength in Numbers." *Nature* 506 (February 2014): 422–423.

Barany, Michael J. "Mathematical Research in Context." Master's thesis, University of Edinburgh, Edinburgh, 2010.

Barany, Michael J., and Donald MacKenzie. "Chalk: Materials and Concepts in Mathematics Research." *Representation in Scientific Practice Revisited* (2014): 107–130.

———. "A Dusty Discipline." In *The Best Writing on Mathematics 2015*, edited by Mircea Pitici, 1–6. Princeton, NJ: Princeton University Press, 2016.

Becker, Daniel Levin. *Many Subtle Channels: In Praise of Potential Literature*. Boston: Harvard University Press, 2012.

Berggren, John Lennart. *Episodes in the Mathematics of Medieval Islam*. New York: Springer, 1986.

Beziau, Jean-Yves, Mihir Chakraborty, and Soma Dutta. *New Directions in Paraconsistent Logic*. New Delhi: Springer, 2014.

Bieberbach, Ludwig. "Stilarten mathematischen Schaffens." *Sitzungsbericht der preußischen Akademie der Wissenschaften* (1934): 351–360.

Boas, Ralph P. "Can We Make Mathematics Intelligible?" *The American Mathematical Monthly* 88, no. 10 (December 1981): 727–731.

Bobzien, Susanne. "Ancient Logic." In *The Stanford Encyclopedia of Philosophy*, Spring 2014, edited by Edward N. Zalta. https://platc.stanford.edu/archives/spr2014/entries/logic-ancient/.

Borovik, Alexandre. *Mathematics Under the Microscope: Notes on Cognitive Aspects of Mathematical Practice*. Providence, RI: American Mathematical Society, 2010.

Bourbaki, Nicholas. "The Architecture of Mathematics." Translated by Arnold Dresden. *The American Mathematical Monthly* 57, no. 4 (April 1950): 221–232.

Boute, Raymond T. "How to Calculate Proofs: Bridging the Cultural Divide." *Notices of the AMS* 60, no. 2 (2013).

Brouwer, Luitzen Egbertus Jan, and Dirk van Dalen. *Brouwer's Cambridge Lectures on Intuitionism*. Cambridge, UK: Cambridge University Press, 2011.

Brown, Ezra, and Jason C. Brunson. "Fibonacci's Forgotten Number." *The College Mathematics Journal* 39, no. 2 (2008) 112–120.

Brown, Kevin S. *Galileo's Anagrams and the Moons of Mars*. http://www.mathpages.com/home/kmath151/kmath151.htm. Accessed February 16, 2016.

Byrne, Oliver. *The First Six Books of the Elements of Euclid: In Which Coloured Diagrams and Symbols Are Used Instead of Letters for the Greater Ease of Learners*. London: William Pickering, 1847.

Cajori, Florian. "Historical Note on the Newton-Raphson Method of Approximation." *The American Mathematical Monthly* 18, no. 2 (1911): 29–32.

———. "Origin of the Name 'Mathematical Induction.'" *The American Mathematical Monthly* 25, no. 5 (1918): 197–201.

Calvino, Italo. *Six Memos for the Next Millennium*. Translated by Geoffrey Brock. Boston: Mariner Books, 1988.

Campbell-Kelly, Martin. *The History of Mathematical Tables: From Sumer to Spreadsheets*. Oxford: Oxford University Press, 2003.

Cardano, Girolamo. *Ars Magna or the Rules of Algebra*. Translated by T. Richard Witmer. Mineola, NY: Dover Publications, Inc., 1968.

———. *Hieronymi Cardani, Praestantissimi Mathematici, Philosophi, Ac Medici, Artis Magnae Sive De Regulis Algebraicis Lib. unus: qui & totius operis de arithmetica, quod opus perfectum inscribitur, est in ordine decimus*. SMITH 512 1545 C17. Norimbergae: Petreius, 1545.

Carroll, Lewis. *Alice in Wonderland (Norton Critical Editions)*. New York: WW Norton & Company, 2013.

Carroll, Lewis, and Martin Gardner. *The Annotated Alice: The Definitive Edition*. London: Penguin, 2001.

Cavendar, Anna, Daniel S. Otero, Jeffrey P. Bigham, and Richard E. Ladner. "ASL-STEM Forum: Enabling Sign Language to Grow through Online Collaboration". In *Proceedings of 28th ACM CHI 2010 Conference on Human Factors in Computing Systems* (2010): 2075–2078. http://doi.acm.org/10.1145/1753326.1753692.

Chang, Kenneth. "A Possible Breakthrough in Explaining a Mathematical Riddle." *The New York Times*, September 2012.

———. "Marina Ratner, Émigré Mathematician Who Found Midlife Acclaim, Dies at 78." *The New York Times*, July 2017.

Chevalley, Claude. "Variations du style mathématique." *Revue de Métaphysique et de Morale* (1935): 375–384.

Clay Mathematics Institute. *Euclid's Elements*. http://www.claymath.org/euclids-elements. Accessed February 23, 2016.

Cohen, Joel S. *Computer Algebra and Symbolic Computation: Mathematical Methods*. Natick, MA: A K Peters, 2003.

Columbia University, Department of Mathematics. *What PhD Graduates are Assumed to Know*. http://www.math.columbia.edu/programs-math/graduate-program/what-graduate-students-are-assumed-to-know/. Accessed July 27, 2016.

Conway, John H., and Joseph Shipman. "Extreme Proofs I: The Irrationality of $\sqrt{2}$." In *The Best Writing on Mathematics 2014*, edited by Mircea Pitici, 216–227. Princeton, NJ: Princeton University Press, 2015.

Cullen, Christopher. *Astronomy and Mathematics in Ancient China: The 'Zhou Bi Suan Jing'*. Needham Research Institute Studies, Vol. 1. Cambridge University Press, 2006.

Cundy, Henry Martyn, and Arthur Percy Rollett. *Mathematical Models*. 3rd ed. St. Albans, UK: Tarquin, 1981.

Davis, Philip J., and Reuben Hersh. *The Mathematical Experience*. Boston: Birkhäuser, 1981.

Decker, Rick. *Swatting flies with a sledgehammer*. July 19, 2012. https://math.stackexchange.com/questions/172509/swatting-flies-with-a-sledgehammer. Accessed March 14, 2018.

Descartes, René. *The Geometry of Rene Descartes*. Translated by David Eugene Smith and Marcia L. Latham. Chicago: Open Court, 1925.

Devlin, Keith. The Problem with Word Problems. May 2010. https://www.maa.org/external_archive/devlin/devlin_05_10.html. Accessed July 19, 2017.

Dieudonné, Jean. *History of Algebraic Geometry*. Translated by Judith D. Sally. Monterey, Calif.: Wadsworth, 1985.

Doyle, Arthur Conan. *The Memoirs of Sherlock Holmes*. Mineola, NY: Dover, 2010.

Doyle, Tim, Lauren Kitler, Robin Miller, and Albert Schueller. *Proofs Without Words and Beyond*. August 2014. http://www.maa.org/publications/periodicals/convergence/proofs-without-words-and-beyond. Accessed June 16, 2017.

Duchêne, Ludmila, and Agnès Leblanc. *Rationnel mon Q*. Paris: Hermann, 2010.

Dudley, Underwood. "What is Mathematics For?" *Notices of the American Mathematical Society* 57, no. 5 (May 2010): 608–613.

Editorial. "Parallel Lines." *Nature* 506, no. 7489 (2014): 407–408.

Eisenman, Peter. "From Object to Relationship II: Casa Giuliani Frigerio: Giuseppe Terragni Casa Del Fascio." *Perspecta* 13/14 (1971): 36–65.

Ellisllk. Oumathpo. http://ellisllk.lautre.net/mathematique/oumathpo/oumathpo.html. Accessed July 27, 2017.

Emch, Arnold. "New Models for the Solution of Quadratic and Cubic Equations." *National Mathematics Magazine* 9, no. 6 (1935): 162–164.

Ewald, William Bragg. *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*. Vol 2. Oxford: Oxford University Press, 1996.

Fadiman, Clifton. *Fantasia Mathematica*. New York: Simon & Schuster, 1958.

Fauvel, John, and Jeremy Gray. *The History of Mathematics: A Reader*. Basingstoke, UK: Palgrave Macmillan Education, 1987.

Feynman, Richard. *What Do You Care What Other People Think?* New York: Norton, 1988.

Fibonacci, Leonardo. *Fibonacci's Liber Abaci: A Translation Into Modern English of Leonardo Pisano's Book of Calculation*. Translated by Laurence Sigler. New York: Springer-Verlag, 2003.

Fine, Benjamin, and Gerhard Rosenberger. *The Fundamental Theorem of Algebra*. New York: Springer Verlag, 1997.

Frame, J. Sutherland. "Machines for Solving Algebraic Equations." *Mathematics of Computation* 1, no. 9 (1945): 337–353.

Galileo Galilei. *Discoveries and Opinions of Galileo*. Translated by Stillman Drake. New York: Anchor Books, 1957.

———. *Sidereus Nuncius, or The Sidereal Messenger*, 2nd ed. Translated with commentary by Albert Van Helden. Chicago: University of Chicago Press, 2015.

Gardner, Martin. "Mathematical Games: 'Look-see' proofs that offer visual proof of complex algebraic formulas." *Scientific American* 229, no. 4 (October 1973): 114–118.

Gentzen, Gerhard. “Untersuchungen über das logische Schließen (Investigations into Logical Inference).” *Mathematische zeitschrift* 39, no. 1 (1935): 176–210.

Ghys, Étienne. “Inner Simplicity vs. Outer Simplicity.” In *Simplicity: Ideals of Practice in Mathematics and the Arts*, edited by Roman Kossak and Philip Ording, 1–14. New York: Springer, 2017.

Gilbreth, Frank Bunker, and Lillian Moller Gilbreth. “Process Charts—First Steps in Finding the One Best Way.” New York: American Society of Mechanical Engineers (ASME), 1921.

Glaser, Paul Lee. “A Study of Perceptions of Mathematical Signs: Implications for Teaching.” Master’s thesis, National Institute for the Deaf, Rochester Institute of Technology, 2005.

Gleason, Andrew. “Angle Trisection, the Heptagon, and the Triskaidecagon.” *American Mathematical Monthly* 95, no. 3 (March 1988): 185–194.

Goethe, Johann Wolfgang von. *Maxims and Reflections*. Translated by Elisabeth Stopp. London: Penguin, 1998.

Goldman, Corrie. Inside a mathematical proof lies literature, says Stanford’s Reviel Netz. May 7, 2012. http://news.stanford.edu/news/2012/may/math-literature-netz-050712.html. Accessed February 17, 2018.

Gowers, Timothy. *Discovering a Formula for the Cubic*. 2007. https://gowers.wordpress.com/2007/09/15/discovering-a-formula-for-the-cubic/. Accessed June 1, 2016.

———. Is Massively Collaborative Mathematics Possible? 2009. https://gowers.wordpress.com/2009/01/27/is-massively-collaborative-mathematics-possible/. Accessed February 19, 2016.

Gowers, Timothy, and Michael Nielsen. “Massively Collaborative Mathematics.” *Nature* 461, no. 7266 (2009): 879–881.

Grant, Edward. *A Source Book in Medieval Science*. Cambridge, MA: Harvard University Press, 1974.

Grattan-Guinness, Ivor. *The Norton History of the Mathematical Sciences: The Rainbow of Mathematics*. London: WW Norton & Company, 1997.

———. *The Search for Mathematical Roots, 1870–1940: Logics, Set Theories and the Foundations of Mathematics from Cantor through Russell to Godel*. Princeton, NJ: Princeton University Press, 2011.

Grosholz, Emily R. *Representation and Productive Ambiguity in Mathematics and the Sciences*. Oxford: Oxford University Press, 2007.

Gutman, Kellie O. “Quando Che’l Cubo.” *Mathematical Intelligencer* 27, no. 1 (2005): 32–36.

Halliday, David, Jearl Walker, and Robert Resnick. *Fundamentals of Physics Extended*, 5th ed. New York: John Wiley & Sons, 1997.

Halmos, Paul Richard. *I Want to be a Mathematician: An Automathography*. New York: Springer-Verlag, 1985.

Haran, Brady. *The Shortest Papers Ever: Numberphile*. December 2016. https://www.youtube.com/watch?v=QvvkJT8myeI. Accessed February 3, 2018.

Harris, Michael. *Mathematics without Apologies: Portrait of a Problematic Vocation*. Princeton, NJ: Princeton University Press, 2015.

Heath, Thomas Little. *The* Method *of Archimedes*. Cambridge, UK: Cambridge University Press, 1912.

Heath, Thomas Little, et al. *Apollonius of Perga: Treatise on Conic Sections with Introductions Including an Essay on Earlier History on the Subject*. Cambridge, UK: Cambridge University Press, 1896.

———. *The Thirteen Books of Euclid's Elements*. Cambridge, UK: Cambridge University Press, 1926.

Henderson, David W. *Geometric Solutions of Quadratic and Cubic Equations*. http://www.math.cornell.edu/~dwh/papers/geomsolu/geomsolu.html. Accessed August 8, 2017.

Herbst, Patricio G. "Establishing a Custom of Proving in American School Geometry: Evolution of the Two-Column Proof in the Early Twentieth Century." *Educational Studies in Mathematics* 49, no. 3 (2002): 283–312.

Herstein, Israel N. *Topics in Algebra*, 2nd ed. New York: John Wiley & Sons, 1975.

Heyting, Arend. *Intuitionism: An Introduction*. Amsterdam: North-Holland, 1956.

Higgins, T. J. "Slide-Rule Solutions of Quadratic and Cubic Equations." *The American Mathematical Monthly* 44, no. 10 (1937): 646–647.

Hilbert, David, Paul Bernays, and Leo Unger. *The Foundations of Geometry*. La Salle, IL: Open Court, 1992.

Hoffman, Paul. *The Man Who Loved Only Numbers: The Story of Paul Erdős and the Search for Mathematical Truth*. New York: Hyperion, 1998.

Høyrup, Jens. "The Babylonian Cellar Text BM 85200 + VAT 6599 Retranslation and Analysis". In *Amphora: Geburtstag Festschrift for Hans Wussing on the Occasion of his 65th Birthday*, edited by Sergei S. Demidov, Menso Folkerts, David E. Rowe, and Christoph J. Scriba, 315–358. Basel: Birkhäuser, 1992.

Hull, Thomas C. "Solving Cubics with Creases: The Work of Beloch and Lill." *American Mathematical Monthly* 118, no. 4 (2011): 307–315.

Hurd, Cuthbert C. "A Note on Early Monte Carlo Computations and Scientific Meetings." *IEEE Annals of the History of Computing*, no. 2 (1985): 141–155.

Huygens, Christiaan. "De Saturni Luna Observatio Nova." *Oeuvres Completes de Christiaan Huygens* 15 (1656), 172–177. La Haye: M. Nijhoff,1925.

Iemhoff, Rosalie. "Intuitionism in the Philosophy of Mathematics." In *The Stanford Encyclopedia of Philosophy*, Winter 2016, edited by Edward N. Zalta. Metaphysics Research Lab, Stanford University. https://plato.Stanford.edu/entries/intuitionism.

Ifrah, Georges. *The Universal History of Numbers*. Translated by David Bellos, E. F. Harding, Sophie Wood, and Ian Monk. New York: John Wiley & Sons, 2000.

Isaacs, Rufus. "Two Mathematical Papers Without Words." *Mathematics Magazine* 48, no. 4 (September 1975): 198.

Jackson, Allyn. "Teaching Math in America: An Exhibit at the Smithsonian." *Notices of the AMS* 49, no. 9 (October 2002): 1082–1083.

Jensen, Shane Tyler. "The Laguerre-Samuelson Inequality with Extensions and Applications in Statistics and Matrix Theory." M. Sc., Department of Mathematics and Statistics, McGill University, 1999.

Johnson, Jesse. *Debunking knot theory's favourite urban legend*. November 7, 2013. https://ldtopology.wordpress.com/2013/11/07/debunking-knot-theorys-favourite-urban-legend/. Accessed January 12, 2018.

Kalman, Dan. *Uncommon Mathematical Excursions: Polynomia and Related Realms*. Dolciani Mathematical Expositions 35. Washington D.C.: Mathematical Association of America, 2009.

Kalman, Dan, and Maurice Burke. "Solving Cubic Equations with Curly Roots." *Mathematics Teacher* 108, no. 5 (2015): 392–397.

Khayyam, Omar. *Algebra wa Al-Muqabala: An Essay by the Uniquely Wise 'Abel Fath Omar Bin Al-Khayyam on Algebra and Equations*. Translated by Roshdi Khalil. Reading, UK: Garnet Publishing, 2008.

Khayyam, Omar. *The Algebra of Omar Khayyam*. Translated by Daoud S. Kasir. New York: Columbia Teachers College, 1931.

Kolata, Gina. "At Last, Shout of 'Eureka!' in Age-Old Math Mystery." *New York Times*, June 24, 1993.

Krantz, Steven G. *The Proof is in the Pudding: A Look at the Changing Nature of Mathematical Proof*. New York: Springer, 2007.

Lacey, Laurie L. Flowcharting Proofs. http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/flowcharting-proofs. Accessed April 11, 2016.

Lagrange, Joseph-Louis de, and Joseph-Alfred Serret. *Oeuvres de Lagrange*. Paris: Gauthier-Villars, 1870.

Lakatos, Imre, John Worrall, and Elie Zahar, eds. *Proofs and Refutations: The Logic of Mathematical Discovery*. Cambridge, UK: Cambridge University Press, 1976.

Lakoff, George, and Rafael Núñez. *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being*. New York: Basic Books, 2000.

Lamb, Evelyn. "Two-Hundred-Terabyte Maths Proof is Largest Ever." *Nature* 534 (2016): 17–18.

Lamport, Leslie. "How to Write a Proof." *The American Mathematical Monthly* 102, no. 7 (1995): 600–608.

Lander, Leon J., and Thomas R. Parkin. "Counterexample to Euler's conjecture on sums of like powers." *Bulletin of the American Mathematical Society* 72, no. 6 (1966): 1079.

Le Lionnais, François. "Lipo: First Manifesto." In *Oulipo: A Primer of Potential Literature*, edited by Warren Motte, 26–28. Champaign, IL: Dalkey Archive Press, 1986.

Lill, Eduard. "Résolution graphique des équations numériques de tous les degrés à une seule inconnue, et description d'un instrument inventé dans ce but." *Nouvelles annales de mathématiques, journal des candidats aux écoles polytechnique et normale* 6 (1867): 359–362.

Linderholm, Carl E. *Mathematics Made Difficult*. London: Wolfe, 1971.

Littlewood, John Edensor, and Béla Bollobás. *Littlewood's Miscellany*. Cambridge, UK: Cambridge University Press, 1986.

Lodder, Jerry. *Deduction through the Ages: A History of Truth*. https://www.maa.org/press/periodicals/convergence/deduction-through-the-ages-a-history-of-truth. Accessed August 8, 2017.

Mac Lane, Saunders. "Van der Waerden's *Modern Algebra*." *Notices of the American Mathematical Society* 44, no. 3 (March 1997): 321–322.

Mancosu, Paolo. Mathematical style. 2010. https://plato.stanford.edu/archives/spr2010/entries/mathematical-style/. Edward N. Zalta (ed.). Accessed August 8, 2017.

Matthews, Harry, and Alastair Brotchie, eds. *Oulipo Compendium*. London: Atlas Press, 1998.

Mazur, Barry. "*History of Mathematics* as a Tool." 2013. http://www.math.harvard.edu/~mazur/papers/History.tool.pdf. Accessed August 3, 2017.

McFarland, Matt. "The Unstoppable TI-84 Plus: How an Outdated Calculator Still Holds a Monopoly on Classrooms." *The Washington Post*. September 2, 2014.

Mermin, N. David. "E Pluribus Boojum: The Physicist as Neologist." *Physics Today* 34, no. 4 (1981): 46.

Metropolis, Nicholas, and Stanislaw Ulam. “The Monte Carlo Method.” *Journal of the American Statistical Association* 44, no. 247 (1949): 335–341.

Mixedmath. *Mathematics Stack Exchange*. 2011. http://math.stackexchange.com/questions/56603/provenance-of-hilbert-quote-on-table-chair-beer-mug. Accessed February 3, 2018.

Morrison, Philip. “Fermi Questions.” *American Journal of Physics* 31, no. 8 (1963): 626–627.

Motion Picture Arts and Sciences, Academy of. *Screenwriting Resources*. 2017. http://www.oscars.org/nicholl/screenwriting-resources. Accessed July 25, 2017.

Nathanson, Melvyn B. “Desperately Seeking Mathematical Truth.” *Notices of the AMS* 55, no. 7 (2008): 773.

National Air and Space Museum. *Slide Rule, 5-Inch, Pickett N600-ES, Apollo 13*. https://airandspace.si.edu/collection-objects/slide-rule-5-inch-pickett-n600-es-apollo-13. Accessed November 3, 2017.

Nelson, David, ed. *The Penguin Dictionary of Mathematics*, 4th ed. London: Penguin, 2008.

New York State Education Department, Office of State Assessment. *Past Examinations*. http://www.nysedregents.org. Accessed April 20, 2015.

Ng, Tuen Wai. *Lagrange’s Work on General Solution Formulae for Polynomial Equations*. http://hkumath.hku.hk/course/MATH2001/MATH2001(2012lecture3334).pdf, 2012. Accessed January 28, 2017.

Nickerson, James. Private correspondence, October, 2015.

Nogrady, Henry A. “A New Method for the Solution of Cubic Equations.” *American Mathematical Monthly* 44, no. 1 (1937): 36–38.

Nordgaard, Martin A. “Sidelights on the Cardan-Tartaglia Controversy.” *National Mathematics Magazine* 12, no. 7 (1938): 327–346.

O’Connor, John J., and Edmund F. Robertson. *The MacTutor History of Mathematics Archive: An Overview of Babylonian Mathematics.* 2000. http://www-history.mcs.st-andrews.ac.uk/HistTopics/Babylonian_mathematics.html. Accessed September 18, 2017.

———. *The MacTutor History of Mathematics Archive: Francesco Paolo Cantelli*. 2007. http://www-history.mcs.st-andrews.ac.uk/Biographies/Cantelli.html. Accessed November 12, 2016.

———. *The MacTutor History of Mathematics Archive: François Viète*. 2000. http://www-history.mcs.st-andrews.ac.uk/Biographies/Viete.html. Accessed October 9, 2015.

———. *The MacTutor History of Mathematics Archive: Leonhard Euler*. http://www-history.mcs.st-and.ac.uk/Biographies/Euler.html. Accessed July 9, 2017.

Oulipo. *Un Certain Disparate: Entretiens avec François Le Lionnais.* October 10, 2010. http://blogs.oulipo.net/fll/. Accessed May 6, 2017.

Papakonstantinou, Joanna M., and Richard A. Tapia. “Origin and Evolution of the Secant Method in One Dimension.” *The American Mathematical Monthly* 120, no. 6 (2013): pp. 500–518.

Patent and Trademark Office, United States. *United States Patent Classification Class Numbers and Titles*. https://www.uspto.gov/web/patents/classification/selectnumwithtitle.htm. Accessed November 23, 2016.

Peano, Giuseppe, and Hubert C. Kennedy. *Selected Works of Giuseppe Peano*. Toronto: University of Toronto Press, 1973.

Pétard, H. “A Contribution to the Mathematical Theory of Big Game Hunting.” *American Mathematical Monthly* 45, no. 7 (August 1938): 446–447.

Poincaré, Henri. *The Value of Science: Essential Writings of Henri Poincaré*. New York: Modern Library, 2001.

Pólya, Georg. *How to Solve It: A New Aspect of Mathematical Method*. Princeton, NJ: Princeton University Press, 1945.

Queneau, Raymond. *Les Fondements de la littérature: d'après David Hilbert*. Vol. 3. Paris: La Bibliothèque oulipienne, 1976.

———. "The Place of Mathematics in the Classification of the Sciences." In *Letters, Numbers, Forms: Essays 1928–1970*. Translated by Jordan Stump. Urbana, IL: University of Illinois Press, 2007.

Rabouin, David. "Styles in Mathematical Practice." In *Cultures Without Culturalism: The Making of Scientific Knowledge*, edited by Karine Chemla and Evelyn Fox Keller. Durham, NC: Duke University Press, 2017.

Rashed, Roshdi. *The Development of Arab Mathematics: Between Arithmetic and Algebra*. Translated by A. F. W. Armstrong. Dordrecht: Springer Science & Business Media, 1994.

Rawles, Bruce. *Sacred Geometry Introductory Tutorial*. 2017. https://www.geometrycode.com/sacred-geometry/. Accessed July 31, 2017.

Regazzini, Eugenio. "Probability and Statistics in Italy During the First World War I. Cantelli and the Laws of Large Numbers." *J. Électron. Hist. Probab. Stat* 1 (2005): 1–12.

Riley, Kenneth Franklin, Michael Paul Hobson, and Stephen John Bence. *Mathematical Methods for Physics and Engineering: A Comprehensive Guide*. Cambridge, UK: Cambridge University Press, 1998.

Robbins, Martin. *This Is a News Website Article About a Scientific Paper*. 2010. https://www.theguardian.com/science/the-lay-scientist/2010/sep/24/1. Accessed May 6, 2015.

Roberts, Siobhan. *Genius at Play: The Curious Mind of John Horton Conway*. New York: Bloomsbury Publishing USA, 2015.

Robinson, Sara. "Russian Reports He Has Solved a Celebrated Math Problem." *The New York Times*. April 15, 2003.

Rota, Gian-Carlo. *Indiscrete Thoughts*. Boston: Birkhäuser Boston, 2008.

Salzer, Herbert E., Charles H. Richards, and Isabelle Arsham. *Table for the Solution of Cubic Equations*. New York: McGraw-Hill, 1958.

Samuelson, Paul A. "How Deviant Can You Be?" *Journal of the American Statistical Association* 63, no. 324 (1968): 1522–1525.

Sangwin, Chris. J. "Modelling the Journey from Elementary Word Problems to Mathematical Research." *Notices of the American Mathematical Society* 58. no. 10 (November 2011): 1436–1445.

Senechal, Marjorie. "Mathematical Communities: The Continuing Silence of Bourbaki—An Interview with Pierre Cartier, June 18, 1997." *The Mathematical Intelligencer* 20, no. 1 (1998): 22–28.

Serre, Jean-Pierre. *How to Write Mathematics Badly*. November 10, 2003. https://www.youtube.com/watch?v=ECQyFzzBHlo. Accessed August 24, 2017.

Sharp, John. *Surfaces: Explorations with Sliceforms*. St. Albans, UK: Tarquin, 2004.

Shen, Kangshen, John N. Crossley, Anthony Wah-Cheung Lun, and Hui Liu. *The nine chapters on the mathematical art: Companion and commentary*. Oxford, England: Oxford University Press, 1999.

Sleeman, Derek H. "Solving Linear Algebraic Equations." *Mathematics in School* 13, no. 4 (1984): 37–38.

Slupinski, Marcus J., and Robert J. Stanton. "The Special Symplectic Structure of Binary Cubics." In *Representation Theory, Complex Analysis, and Integral Geometry*, edited by Bernhard Krötz, Omer Offen, and Eitan Sayag, 185–230. New York: Springer, 2012.

Smith, William P. "Editor's Note." *The American Statistician* 34, no. 251. (1980).

Sobel, Dava. *Galileo's Daughter: A Historical Memoir of Science, Faith and Love*. New York: Bloomsbury Publishing USA, 1999.

Stedall, Jacqueline A. *From Cardano's Great Art to Lagrange's Reflections: Filling a Gap in the History of Algebra*. Vol. 5. Zürich: European Mathematical Society, 2011.

Stillwell, John. *Mathematics and Its History (Undergraduate Texts in Mathematics)*. New York: Springer, 1989.

Tardy, Christine. "The Role of English in Scientific Communication: Lingua Franca or Tyrannosaurus Rex?" *Journal of English for Academic Purposes* 3, no. 3 (2004): 247–269.

Tartaglia, Niccolò. *Quesiti et inventioni diverse*. Edited by A. Masoti. Brescia: Ateneo di Brescia, 1546. http://dx.doi.org/10.3931/e-rara-9183

Thompson, Clive. "The Year in Ideas: Outsider Math." *The New York Times*, December 15, 2002.

Thompson, Robert C. "Author vs. Referee: A Case History for Middle Level Mathematicians." *The American Mathematical Monthly* 90, no. 10 (Dec. 1983): 661–668.

Thurston, William P. "On Proof and Progress in Mathematics." *Bull. Amer. Math. Soc.* 30 (1994): 161–177.

van der Waerden, Bartel L. *Algebra*. Vol. 1. New York: Springer-Verlag, 1970.

van der Waerden, Bartel L., Emil Artin, Emmy Noether, and Fred Blum. *Moderne Algebra*. Berlin: Springer, 1937.

Villani, Cédric. *Birth of a Theorem: A Mathematical Adventure*. New York: Farrar, Straus & Giroux, 2015.

Walster, G. William, and Eldon R. Hansen. *Solving a Nonlinear Equation Through Intervalic Arithmetic and Term Consistency*, US Patent 6,823,352 filed September 13, 2001, and issued November 23, 2004.

Whitehead, Alfred North, and Bertrand Russell. *Principia Mathematica to *56*. London: Cambridge University Press, 1970.

Wiedijk, Freek, ed. *The Seventeen Provers of the World*. New York: Springer, 2006.

Wikipedia. *Texas Instruments signing key controversy*. https://en.wikipedia.org/wiki/Texas_Instruments_signing_key_controversy. Accessed August 2, 2017.

Wilkinson, James H. "The Perfidious Polynomial." *Studies in Numerical Analysis* 24 (1984): 1–28.

Woodward, Chris. *Mathoverflow.* August 25, 2010. https://mathoverflow.net/questions/36596/refereeing-a-paper. Accessed July 14, 2017.

You, Jia. "Who Are the Science Stars of Twitter?" *Science* 345, no. 6203 (2014): 1440–1441.

Zeilberger, Doron. *Dr. Z's Opinions*. 2018. http://sites.math.rutgers.edu/~zeilberg/OPINIONS.html. Accessed February 3, 2018.

Zukav, Gary. *The Dancing Wu Li Masters*. New York: Bantam Books, 1980.

Index

Abraham, Ralph, 230
ALGORITHMIC, 63
ANALYTIC, 95
ANCIENT, 35
ANOTHER COUNTEREXAMPLE, 73
ANOTHER INTERPRETED, 175
ANOTHER MEDIEVAL, 163
ANOTHER SUBSTITUTION, 203
ANOTHER SYMMETRY, 53
ANOTHER TRANSLATED, 173
ANTIQUITY, 121
Appel, Kenneth, 114
APPROXIMATE, 157
ARBORESCENT, 129
Archimedes, 104, 192, 208
Artin, Emil, 144
arXiv, 86
Ashbery, John, 234
ASL (American Sign Language), *see* ANOTHER INTERPRETED
Audin, Michèle, ix, 172
AUDITORY, 61
AUTHORITY, 223
AXIOMATIC, 13
AXONOMETRIC, 145

BACK OF THE ENVELOPE, 149
Barany, Michael, 48
Beloch, Margharita Piazolla, 90
Berge, Claude, ix, 84
Berggren, Len, 165
Bhāskara II, 160
Bieberbach, Ludwig, 174
BLACKBOARD, 47
BLOG, 167
Boas, Ralph, 222
Bolzano, Bernard, 96
Borovik, Alexandre, 90
Bourbaki, Nicolas, 15, 222
Boute, Raymond, 40
Braffort, Paul, 84
Brouwer, L. E. J., 30, 190
Byrne, Oliver, 118

Cajori, Florian, 92
CALCULATOR, 135
CALCULUS, 75
Calvino, Italo, ix, x
Cantelli, Paolo, 188
Cardano's formula, 70
Cardano, Girolamo, 18, 20, 72, 83, 85, 93, 97, 103, 104, 108, 218
Carroll, Lewis (Charles Lutwidge Dodgson), 194, 232
Chevalley, Claude, 96
CHROMATIC, 117
CLEVER, 111
Cohen, Joel, 64
color, *see* CHROMATIC
Colvis, John P., 115
COMPUTER ASSISTED, 113
CONTRAPOSITIVE, 31
Conway, John H., 20, 22
CORRESPONDENCE, 197
COUNTEREXAMPLE, 71
cubic equation, 2
CUTE, 109

d'Alembert's theorem, 171
da Coi, Zuanne de Tonini, 93
da Vinci, Leonardo, 93
Dante Alighieri, 194
Davis, Philip, 249
DEFINITIONAL, 43
del Ferro, Scipione, 18, 83, 85, 104
Delsarte, Jean, 239
Descartes, René, 106
DIALOGUE, 209
DOGGEREL, 193
Doyle, Tim, 24
du Sautoy, Marcus, 84
Duchêne, Ludmila, x
Duchamp, Marcel, ix
Dudley, Underwood, 12
Duhem, Pierre, 170

economy, 4, 10
eigenvalue, 33
Eisenman, Peter, 148
elchataym, 78
ELECTROSTATIC, 227
ELEMENTARY, 9

Emch, Arnold, 68
Erdős, Paul, 245, 251
Euclid, 124
Euclidean algorithm, 64
Euler, Leonhard, 224, 226
EXAM, 25
EXHAUSTION, 201
EXPERIMENTAL, 183

Fermat, Pierre de, 92
Fermi, Enrico, 150
Ferrari, Ludovico, 97–104
Feynman, Richard, 118
Fibonacci (Leonardo of Pisa), 78
FIRST PERSON, 225
FLOW CHART, 65
FORMULAIC, 69
FOUND, 17
French, *see* TRANSLATED

Galileo, ix, x, 192, 218
Gentzen, Gerhard, 130
GEOMETRIC, 141
German, *see* ANOTHER TRANSLATION
Ghys, Étienne, 8, 88
Goethe, 50
Gowers, Timothy, 60, 212
Grosholz, Emily, 2

Haken, Wolfgang, 114
Hales, Thomas, 114
Halmos, Paul, 4, 184
HAND WAVING, 155
Harris, Michael, 250
Herbst, Patricio, 6
Hersh, Reuben, 239, 249
Heule, Marijn, 202
Hilbert, David, 15, 205
Holmes, Sherlock, 106
homophonic translation, *see* MONDEGREEN
Horner, William, 142
Høyrup, Jens, 38
Humpty Dumpty, 232
Huygens, Christiaan, 192

ILLUSTRATED, 7
INCONSISTENCY, 195
INDENTED, 39
INDUCTION, 91
INTERIOR MONOLOGUE, 213
INTERPRETED, 37
INTUITIONIST, 189
INVENTORS PARADOX, 137

JARGON, 41

Kalman, Dan, 222
Kepler, Johannes, ix, 192
Khayyam, Omar, 85, 106, 165
Klein, Felix, 228
Knuth, Donald, 81
Kossak, Roman, 197, 198
Kreisel, Georg, 84
Kullmann, Oliver, 202

Laba, Izabella, 170
Ladner, Richard, 180
Lagrange, Joseph-Louis, 55
Lakatos, Imre, 60, 74, 124, 214
Lakoff, George, 156
Lamport, Leslie, 40
larding, 240
Le Lionnais, François, ix, 84
Leblanc, Agnès, x, 81
Lill, Eduard, 252
Lill's method, 90
Lill, Eduard, 142
Lindholm, Carl, 212
Littlewood, J. E., 230
Łukasiewicz, Jan, 132

Mac Lane, Saunders, 144
MacKenzie, Donald, 48
Marek, Victor, 202
Marey, Étienne-Jules, 156
MARGINALIA, 125
MATRICES, 33
Mazur, Barry, 36
McCleary, John, x, 84
MECHANICAL, 207

MEDIEVAL, 77
Mermin, David, 222
metaphor, 50
Metropolis, Nick, 186
MODEL, 67
MODERN, 143
MONDEGREEN, 231
MONOSYLLABIC, 21
MONTE CARLO, 185
music, *see* AUDITORY
MYSTICAL, 217

Nathanson, Melvyn, 224
NEOLOGISM, 221
Netz, Reviel, 50
NEWSPRINT, 93
Newton's method, 183
Nickerson, James, 180
Noether, Emmy, 15, 144
Nogrady, H. A., 200
Núñez, Rafael, 156

Oldenburg, Henry, 192
OMITTED, 1
OMITTED WITH CONDESCENSION, 105
ONE-LINE, 3
OPEN COLLABORATIVE, 57
Oresme, Nicole, 78
ORIGAMI, 89
Oulipo, ix, 22, 81
Oumathpo, 84
OUTSIDER, 115
Ovid, 231

Pacioli, Luca, 93
PARANOID, 191
PARATAXIS, 87
PATENTED, 139
Paulos, John Allen, 84
Peano, Giuseppe, 15
Perec, Georges, ix
Pétard, H., x, 120
physical model, *see* MODEL
physics, 208, 228
Plato, 93
Poincaré, Henri, 228
Pólya, Georg, 10, 30, 138, 214
Polymath Project, 60
POSTFIX, 133
PREFIX, 131
PREPRINT, 85
PREREQUISITE, 19
PRESCRIBED, 233
prime number theorem, 10
PROBABILISTIC, 187
process chart, *see* FLOW CHART
proof without words, *see* WORDLESS
PSYCHEDELIC, 229
PUZZLE, 11

Queneau, Raymond, ix, x, 22, 84, 142, 205, 235

Rabouin, David, 34
Rashed, Roshti, 170
Ratner, Marina, 88
rebellion, 48
REDUCTIO AD ABSURDUM, 29
REFEREED, 219
RESEARCH SEMINAR, 151
RETROGRADE, 215
Robbins, Martin, 94
Roberts, Siobhan, 22
Rota, Gian-Carlo, 15, 84, 156
Roubaud, Jacques, ix, 84
RPN (reverse Polish notation), 134
RULER AND COMPASS, 27

Samuel, Pierre, 84
Samuelson, Paul, 162
SCREENPLAY, 97
Serre, Jean-Pierre, 8, 224
Shipman, Joseph, 20
sledgehammer, 20
SLIDE RULE, 181
Smith, William P., 188
SOCIAL MEDIA, 83
STATISTICAL, 161
straightedge and compass, *see* RULER AND COMPASS

SUBSTITUTION, 49
Supreme Court, 140
symmetric polynomial, 55
SYMMETRY, 51
symplectic geometry, 153

TABULAR, 199
Tao, Terrence, 60
Tardy, Christine, 174
Tartaglia, Niccolò (Fontana), 18, 83, 93, 97–103, 194
TEA, 153
Terragni, Giuseppe, 148
TeX, 81
Thurston, William, 76, 154
TI-83, *see also* CALCULATOR
TOPOLOGICAL, 119
TRANSLATED, 179
tweet, *see* SOCIAL MEDIA
TWO-COLUMN, 5
TYPESET, 79
Tyson, Neil deGrasse, 84

Ulam, Stanislaw, 84, 186

van der Waerden, B. L., 144, 174
Vasiliev, Nicolai A., 196
VERBAL, 107
Viète, François, 112
Villani, Cédric, 198
violin, 62
von Neumann, John, 212

Wallis, John, 92
Wantzel, Pierre, 94
Weil, Andrè, 239
Wilkinson, James, 158
WORD PROBLEM, 159
WORDLESS, 23

Zeilberger, Doron, 114, 170
Zhang, Yitang, 60